A wave problem in a half-space with a unilateral constraint at the boundary

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On considère l'équation des ondes dans un demi-espacc $\Omega = \{x \in \mathbb{R}^N / x_N > 0\}$:

$$u_{tt} - \Delta u = 0 \qquad \text{dans} \quad Q = \Omega \times]0, \infty[, \tag{1}$$

$$u(x, 0) = u_0(x) \qquad \text{dans} \quad \Omega, \tag{2}$$

$$u_t(x, 0) = u_1(x) \qquad \text{dans} \quad \Omega, \tag{3}$$

avec la condition unilatérale au bord

$$\left.\begin{aligned} u(x', 0, t) &\geqslant 0 \\ \frac{\partial u}{\partial n}(x', 0, t) &\geqslant 0 \\ u(x', 0, t) \cdot \frac{\partial u}{\partial n}(x', 0, t) &= 0 \end{aligned}\right\} \quad \text{sur } \Sigma = \partial\Omega \times]0, \infty[, \tag{4}$$

où n est la normale extérieure à Ω. On ramène ce problème à un problème unilatéral sur le bord, qui fait intervenir l'opérateur A défini par

$$\begin{aligned} u_{tt} - \Delta u &= 0 && \text{dans } Q, \\ u_0 = u_1 &= 0 && \text{dans } \Omega, \\ u(x', 0, t) &= v(x', t) && \text{sur } \Sigma, \end{aligned} \tag{5}$$

et

$$(Av)(x', t) = \frac{\partial u}{\partial n}(x', 0, t). \tag{6}$$

L'opérateur A n'est ni local, ni pseudo-différentiel. Nous montrons cependant que sa restriction à un intervalle de temps borné $]0, T[$ est positive en tant qu'opérateur non borné de $L^2(]0, T[\times \mathbb{R}^{N-1})$ dans lui-même. Nous déduisons alors l'existence et l'unicité pour le problème (1)–(4), avec conservation de l'énergie. La positivité de A défini par (6), qui est une dérivée normale, dépend essentiellement de la géometrie de Ω.

We consider the wave equation in a half-space $\Omega = \{x \in \mathbb{R}^N / x_N > 0\}$:

$$\begin{aligned} u_{tt} - \Delta u &= 0 \qquad && \text{in } Q = \Omega \times (0, \infty), && (1)\\ u(x, 0) &= u_0(x) && \text{in } \Omega, && (2)\\ u_t(x, 0) &= u_1(x) && \text{in } \Omega, && (3) \end{aligned}$$

with the unilateral condition at the boundary

$$\left.\begin{aligned} u(x', 0, t) &\geqslant 0 \\ \frac{\partial u}{\partial n}(x', 0, t) &\geqslant 0 \qquad \text{on } \Sigma = \partial\Omega \times (0, \infty), \\ u(x', 0, t) \cdot \frac{\partial u}{\partial n}(x', 0, t) &= 0 \end{aligned}\right. \tag{4}$$

where n is the exterior normal to Ω. We reduce problem (1)–(4) to a problem on the boundary, which involves the operator A defined by

$$\begin{aligned} u_{tt} - \Delta u &= 0 \qquad && \text{in } Q, \\ u_0 = u_1 &= 0 && \text{in } \Omega, \\ u(x', 0, t) &= v(x', t) && \text{on } \Sigma, \end{aligned} \tag{5}$$

and

$$(Av)(x', t) = \frac{\partial u}{\partial n}(x', 0, t). \tag{6}$$

This operator A is neither local nor pseudo-differential; we show that its restriction to a bounded time interval $(0, T)$ is positive, when it is considered as an unbounded operator from $L^2((0, T) \times \mathbb{R}^{N-1})$ to itself. We deduce that problem (1)–(4) possesses a unique solution, which conserves the energy. The positivity of the normal derivative operator A defined by (6) depends essentially on the geometry of Ω.

1. Introduction

In this paper, we study the following problem: let Ω be a half-space of $\mathbb{R}^N$, defined by

$$\Omega = \{x = (x', x_N) \in \mathbb{R}^N / x_N > 0\}, \tag{1.1}$$

where $x' = (x_1, ..., x_{N-1})$ is the usual notation, and let there be given functions $u_0 \in H^1(\Omega)$ and $u_1 \in L^2(\Omega)$. We assume that $u_0|_{x_N=0}$ is nonnegative, and similarly $-(\partial u_0/\partial x_N)|_{x_N=0}$ (which is, a priori, an element of $H^{-1/2}(\mathbb{R}^{N-1})$) is nonnegative. Later in the article, we set $x' = x$ and $x_N = y$.

We wish to find a solution of the initial and boundary problem

$$u_{tt} - \Delta u \equiv \Box u = f \qquad \text{in } Q_T = \Omega \times (0, T), \tag{1.2}$$

$$u(x, 0) = u_0(x) \qquad \text{in } \Omega, \tag{1.3}$$

$$u_t(x, 0) = u_1(x) \qquad \text{in } \Omega; \tag{1.4}$$

$$\begin{aligned} & u(x', 0, t) \geqslant 0, \\ & -\frac{\partial u}{\partial x_N}(x', 0, t) \geqslant 0, \quad \text{a.e. on } \Sigma_T = \{x/x_N = 0\} \times (0, T) = \partial\Omega \times (0, T), \\ & u(x', 0, t) \frac{\partial u}{\partial x_N}(x', 0, t) = 0. \end{aligned} \tag{1.5}$$

Conditions (1.5) are usually termed unilateral constraints.

Equations (1.2)–(1.5) can be generalized to the case of a slab with, for example, a Dirichlet condition on the other side; thus x belongs to $\mathbb{R}^{N-1} \times (0, L)$ and we have the boundary condition

$$u(x', L, t) = 0, \qquad \forall x', t. \tag{1.5$'$}$$

In case $N = 1$, Eqs. (1.2)–(1.5) and (1.5′) allow one to describe the one-dimensional motions of a slab of linearly elastic material, with zero displacement at $x_3 = L$ and with a unilateral constraint at $x_3 = 0$, i.e., the dynamical situation associated to the Signorini problem [2, 5, 8]. Let us formulate this problem without reference to the specific geometry of a slab: if Ω is the reference configuration, $\Omega \subset \mathbb{R}^3$, if $u = (u_1, u_2, u_3)$ is the displacement vector defined on Ω, and if we choose to work at the approximation of small displacements, the strain tensor ε is defined by

$$\varepsilon_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_i}{\partial x_j}\right). \tag{1.6}$$

As forces are balanced,

$$\sigma_{ij,j} = f_i - \frac{\partial^2 u_i}{\partial t^2}, \tag{1.7}$$

where σ_{ij} is the stress tensor. The constitutive law of an elastic material is

$$\sigma_{ij} = a_{ijkl}(x)\, \varepsilon_{kl}(u) \tag{1.8}$$

and, in the case of an isotropic, homogeneous material,

$$a_{ijkl} = \lambda \delta_{ij}\, \delta_{kl} + \mu(\delta_{ik}\, \delta_{jl} + \delta_{il}\, \delta_{jk}),$$

so that

$$\sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij}. \tag{1.9}$$

Define, for n the exterior unit normal to the boundary of Ω,

$$\begin{aligned} u_N &= u \cdot n \\ u_{iT} &= u_i - u_N n_i \\ u_T &= (u_{iT})_{i=1,2,3} \end{aligned} \tag{1.10}$$

$$\begin{aligned} \sigma_N &= \sigma_{ij} n_i n_j \\ \sigma_{iT} &= \sigma_{ij} n_j - \sigma_N n_i \\ \sigma_T &= (\sigma_{iT})_{i=1,2,3}. \end{aligned} \tag{1.11}$$

Then the Signorini unilateral conditions are given by

$$\left\{ \begin{aligned} u_N &\leqslant 0 \\ \sigma_N &\leqslant 0 \\ u_N \cdot \sigma_N &= 0 \\ \sigma_T &= 0 \end{aligned} \right\} \quad \text{on } \Gamma_S, \tag{1.12}$$

where Γ_S is a given part of the boundary, where the elastic body is not allowed to move outside of its reference configuration. Conditions (1.12) express moreover that whenever $u_N < 0$, the normal strain is zero, because there is no contact between the body and the obstacle (which has the shape of the reference configuration along Γ_S!); and finally, if $u_N = 0$, the reaction of the obstacle is normal to the boundary, and directed inwards.

In the particular case of a slab of homogeneous, isotropic material, with reference configuration $\Omega = \mathbb{R}^2 \times (0, L)$, and a unilateral condition on $\Gamma_S = \mathbb{R}^2 \times \{0\}$, the equations of motion depend only on x_3 and are the following: from (1.6), (1.7) and (1.9),

$$\frac{\partial^2 u_\alpha}{\partial t^2} = \mu \frac{\partial^2 u_\alpha}{\partial x_3^2} + f_\alpha, \qquad \alpha = 1, 2; \tag{1.13}$$

$$\frac{\partial^2 u_3}{\partial t^2} = (\lambda + 2\mu) \frac{\partial^2 u_3}{\partial x_3^2} + f_3. \tag{1.14}$$

The unilateral conditions (1.12) can be translated as follows:

$$n = (0, 0, -1)$$

$$\sigma_N = \sigma_{33} = (\lambda + 2\mu) \frac{\partial u_3}{\partial x_3}$$

$$u_N = -u_3$$

$$\sigma_T = (-\sigma_{13}, -\sigma_{23}, 0) = \left(-\frac{\partial u_1}{\partial x_3}, -\frac{\partial u_2}{\partial x_3}, 0\right)$$

so that

$$\left.\begin{array}{c} u_3 \geqslant 0 \\ (\lambda + 2\mu)\dfrac{\partial u_3}{\partial x_3} \leqslant 0 \\ u_3 \cdot (\lambda + 2\mu)\dfrac{\partial u_3}{\partial x_3} = 0 \\ \dfrac{\partial u_1}{\partial x_3} = \dfrac{\partial u_2}{\partial x_3} = 0. \end{array}\right\} \quad \text{on } \Gamma_S. \tag{1.15}$$

If we choose the condition at $x_3 = L$ to be $u = 0$ for instance, then the system (1.13)–(1.15) uncouples as

$$\begin{array}{ll} \dfrac{\partial^2 u_3}{\partial t^2} = (\lambda + 2\mu)\dfrac{\partial^2 u_3}{\partial x_3^2} + f_3 & \text{on } (0, L) \times (0, T), \\ u_3 \geqslant 0, \quad \dfrac{\partial u_3}{\partial x_3} \leqslant 0, \quad u_3 \dfrac{\partial u_3}{\partial x_3} = 0 & \text{on } \{x = 0\}, \\ u_3 = 0 & \text{on } \{x = L\}, \end{array} \tag{1.16}$$

which is exactly of the form (1.2)–(1.5), (1.5′), and two wave equations on u_1, u_2, with Dirichlet boundary equations.

In the case $N = 2$, one can imagine the following experimental setup which would allow a membrane to vibrate with boundary conditions corresponding to (1.5): let the reference configuration of the membrane be a plane domain Ω, and let $\partial\Omega$ be its boundary. Let $\Gamma_{\mathscr{U}}$ be a part of $\partial\Omega$ where the membrane is tied to a glider, so as to move freely in the transverse direction, and assume that on the remainder of $\partial\Omega$, the membrane is fixed. This realizes Neumann conditions on $\Gamma_{\mathscr{U}}$. If moreover, there is on the glider an obstacle which constraints the membrane to stay on one side of that obstacle, and if the contact between the edge of the membrane and the obstacle does not dissipate energy, then the motion of the membrane would satisfy (1.2)–(1.4), with the boundary condition (1.5) on $\Gamma_{\mathscr{U}}$ and a Dirichlet condition $\partial\Omega\backslash\Gamma_{\mathscr{U}}$.

Nevertheless, the motivation for studying (1.2)–(1.5) does not lie in the mathematical study of the above two situations, which are not very interesting from the mechanical point of view. The real motivation for studying (1.2)–(1.5) is the approach the problem of linear elastic vibrations with unilateral constraints, that is, the general dynamical Signorini problem.

The system of elasticity is much more complicated than the wave equation, but we expect to retain, in this simpler case, some of the features of the more difficult problem of elasticity.

In this paper, we prove that problem (1.2)–(1.5) possesses a unique solution which conserves energy, if we assume that u_0, u_1 and f are smooth enough, viz., $u_0 \in H^{3/2}(\Omega) \cap H_0^1(\Omega)$, $u_1 \in H^{1/2}(\Omega)$ and $f \in H^{3/2}(Q_T)$.

To obtain this result, we reduce problem (1.2)–(1.5) to a problem on the boundary which involves the operator A defined by

$$\begin{aligned} \Box u &= 0 && \text{in } Q_T, \\ u(x, 0) = u_t(x, 0) &= 0 && \text{a.e. on } \Omega, \\ u(x', 0, t) &= v(x', t) && \text{on } \Sigma_T, \end{aligned} \tag{1.17}$$

$$(Av)(x', t) = -\frac{\partial u}{\partial x_N}(x', 0, t). \tag{1.18}$$

In the case $N = 1$, we can express A as a differential operator: if u satisfies (1.6), with $\Omega = (0, +\infty)$, then there exist distributions f and g such that

$$u(x, t) = f(x + t) + g(x - t), \qquad \forall x \geqslant 0, \quad \forall t \geqslant 0.$$

If $u(0, \cdot)$ is in $C^\infty([0, +\infty))$, then we know that the solution u is of class C^∞, and we have

$$\begin{aligned} f(x) + g(x) &= 0 && \text{on } (0, \infty), \\ f'(x) - g'(x) &= 0 && \text{on } (0, \infty), \end{aligned}$$

so that we can choose f and g such that

$$f(x) = g(x) = 0 \qquad \text{on } (0, \infty).$$

Then, the boundary condition,

$$f(t) + g(-t) = v(t), \qquad \forall t \geqslant 0,$$

implies

$$g(-t) = v(t), \qquad \forall t \geqslant 0.$$

Therefore,

$$(Av)(t) = -\frac{\partial u}{\partial x}(0, t) = -f'(t) - g'(-t) = \frac{\partial}{\partial t}(g(-t)) = \frac{\partial v}{\partial t}(t).$$

We can see that in the case $N = 1$, A is just differentiation with respect to time.

If $N \geqslant 2$, this operator is not local; it is not pseudo-differential either, as its symbol is not smooth on an unbounded subset of $\mathbb{R}^N$. We nevertheless use, in Section 3, pseudo-differential techniques to prove that A is positive, when it is considered as an unbounded operator in $L^2((0, T) \times \mathbb{R}^{N-1})$, and we give a number of functional results concerning A.

The reduction of problem (1.1)–(1.5) to a problem on the boundary is performed as follows: if w is the solution of

$$\begin{aligned} \Box w &= f && \text{in } Q_T, \\ w(x, 0) &= w_0(x) && \text{a.e. on } \Omega, \\ w_t(x, 0) &= w_1(x) && \text{a.e. on } \Omega, \\ w(x', 0, t) &= 0 && \text{on } \Sigma_T, \end{aligned} \tag{1.19}$$

we let

$$\varphi(x, t) = \frac{\partial w}{\partial x_N}(x', 0, t)$$

and we show that (1.1)–(1.5) is equivalent to

$$\begin{aligned} Av &\geqslant \varphi \\ v &\geqslant 0 \\ (Av - \varphi)v &= 0 \end{aligned} \tag{1.20}$$

and

$$u = w + v \qquad \text{on the boundary.}$$

Once again, the case $N = 1$ is extremely easy, and one proves immediately that the problem

$$\begin{aligned} \frac{dv}{dt} &\geqslant \varphi \\ v &\geqslant 0 \\ \left(\frac{dv}{dt} - \varphi\right) v &= 0 \\ v(0) &= 0, \end{aligned} \tag{1.21}$$

where $\varphi \in L^1(0, \infty)$, possesses a unique solution which is given explicitly as follows: let

$$\Phi(t) = \int_0^t \varphi(s)\, ds$$

and denote

$$r^- = \max(-r, 0), \qquad \text{for} \quad r \in \mathbb{R}.$$

Then

$$v(t) = \max\{\Phi^-(s)/0 \leqslant s \leqslant t\} + \Phi(t). \tag{1.22}$$

Clearly

$$v(t) \geqslant \Phi^-(t) + \Phi(t) \geqslant 0.$$

Let $\mathscr{U}$ be the open set defined by

$$z(t) = \max\{\Phi^-(s)/0 \leqslant s \leqslant t\}$$
$$\mathscr{U} = \{t \in [0, \infty)/z(t) > \Phi^-(t)\}.$$

Then, on $\mathscr{U}$, z is locally constant, and therefore

$$\frac{dz}{dt}(t) = 0 \qquad \text{a.e. on } \mathscr{U}.$$

On $\mathscr{U}^c$, the complement of $\mathscr{U}$, $z(t) = \Phi^-(t)$; as z is by definition nondecreasing, its derivative, given by

$$\frac{dz}{dt} = \frac{d\Phi^-(t)}{dt} = \begin{cases} -\varphi & \text{if} \quad \Phi(t) < 0, \\ 0 & \text{if} \quad \Phi(t) \geqslant 0, \end{cases}$$

is nonnegative almost everywhere on $\mathscr{U}^c$.

Therefore, we obtain

$$\frac{dv}{dt} = \varphi \qquad \text{on} \quad \mathscr{U} \cup [\mathscr{U}^c \cap \{\Phi(t) \geqslant 0\}],$$

$$\frac{dv}{dt} = 0 \qquad \text{on} \quad \mathscr{U}^c \cap \{\Phi(t) < 0\},$$

where we know that $\varphi \leqslant 0$ on $\mathscr{U}^c \cap \{\Phi(t) < 0\}$.

This proves that v given by (1.21) satisfies (1.22). The uniqueness of $v \in W^{1,1}_{\mathrm{Loc}}(0, \infty)$ is left to the reader.

An alternative way of solving the one-dimensional case is to observe that if we symmetrize u, solution of (1.1)–(1.5) as follows

$$\tilde{u}(x, t) = u(-x, t), \qquad \forall x \leqslant 0,$$

then $\tilde{u}$ has the following properties

$$\Box \tilde{u} = \delta_0 \otimes \nu(t) + f,$$

where the measure ν is given by

$$\langle \nu, \psi \rangle = -2 \int_0^{\infty} \frac{\partial u}{\partial x}(0, t)\, \psi(t)\, dt = 2\langle A[u(0, \cdot)], \psi \rangle$$

so that, according to (1.21),

$$\begin{gathered} \Box \tilde{u} \geqslant f \\ \tilde{u}(\cdot\, t) \geqslant 0 \\ \operatorname{supp}(\Box \tilde{u} - f) \subset \{0\} \times \{t/u(0, t) = 0\}, \end{gathered}$$

which is precisely the formulation of the problem of the string with a pointlike obstacle [7], for which an existence and uniqueness theory has been given in the referenced article.

In Section 4, using techniques from variational inequalities, we show that (1.20), for arbitrary N, possesses a unique solution as soon as φ belongs to the space $L^1((0, T);\ H^{1/2}(\mathbb{R}^{N-1})) \cap W^{1,1}((0, T);\ H^{-1/2}(\mathbb{R}^{N-1}))$. If φ is somewhat smoother, we get the conservation of energy.

Finally, we show that the positivity of A depends on the geometry of the domain, when A is defined by (1.6), and (1.7) is replaced by

$$Av(x', t) = \frac{\partial u}{\partial \nu}(x', t),$$

with ν the exterior normal to Ω.

There seems to be a great difference between unilateral constraints on an open subset of Ω and on a submanifold of lower dimension of Ω.

A clue to the difference between constraints on a lower-dimensional submanifold of $\bar{\Omega}$ and constraints on an open subset of $\bar{\Omega}$ can be found in the comparison of [6] and [7]. In [6] and [7], a vibrating string was constrained to stay above an obstacle; in [6], the obstacle was continuous, and in [7] it was pointlike; one had to impose the conservation of energy in [6] to obtain the uniqueness and, on the contrary, the conservation of energy was a consequence of the equations of motion in [7].

It is possible that this difficulty is not very far from the one which is met in variational inequalities for elliptic operators in a smooth bounded domain Ω. Indeed let

$$K = \{u \in H_0^1(\Omega)/u \geqslant 0\}$$

and

$$L = \{u \in H^1(\Omega)/u|_{\partial\Omega} \geqslant 0\};$$

both problems, for f given in $L^2(\Omega)$,

$$\min_{u \in K} \left(\int |\nabla u|^2 \, dx - \int fu \, dx\right) \tag{1.8}$$

$$\min_{u \in L} \left(\int |\nabla u|^2 \, dx - \int fu \, dx\right) \tag{1.9}$$

admit a unique solution, but it is considerably easier to characterize the solution of (1.9) than that of (1.8).

See Lions [4, Chapter 2.8, Examples 8.1 and 8.2] for more details concerning (1.8) and (1.9).

2. Reduction to a Problem on the Boundary

Let $N \geqslant 1$ be an integer, and let T be positive and bounded; we denote

$$\begin{aligned}
&n = N - 1\\
&\Omega = \mathbb{R}^n \times \mathbb{R}_*^+\\
&Q_T = \Omega \times (0, T), \qquad Q = Q_\infty,\\
&\Gamma = \partial\Omega\\
&\Sigma_T = \Gamma \times (0, T), \qquad \Sigma = \Sigma_\infty;
\end{aligned}$$

the generic point of Ω is denoted by

$$X = (x, y), \qquad x \in \mathbb{R}^n, \, y \in \mathbb{R}_*^+,$$

and the problem we are interested in is

$$u_{tt} - \Delta u = f \qquad \text{in } Q_T, \tag{2.1}$$

$$u(X, 0) = u_0(X) \qquad \text{in } \Omega, \tag{2.2}$$

$$u_t(X, 0) = u_1(X) \qquad \text{in } \Omega, \tag{2.3}$$

$$
\begin{aligned}
& u \geqslant 0 \\
& -\frac{\partial u}{\partial y} \geqslant 0 \qquad \text{on } \Sigma_T. \\
& u \frac{\partial u}{\partial y} = 0
\end{aligned}
\tag{2.4}
$$

We define a boundary operator A as follows: let φ belong to $\mathscr{D}(\mathbb{R}^n \times (0, T))$, and let u be the solution of

$$
\begin{aligned}
\square u &= 0 && \text{in } Q_T, \\
u(X, 0) = u_t(X, 0) &= 0 && \text{in } \Omega, \\
u(x, 0, t) &= \varphi(x, t) && \text{on } \Sigma_T.
\end{aligned}
\tag{2.5}
$$

Then, it is clear that u is of class C^∞ on Q_T, and therefore, $(\partial u/\partial y)(x, 0, t)$ is of class C^∞.

More precisely, if φ is supported in $B(0, R) \times [t_0, T]$ with

$$B(0, R) = \{x \in \mathbb{R}^n / |x| \leqslant R\},$$

then u is supported in

$$\{(x, y, t)/(|x|^2 + y^2)^{1/2} \leqslant R + t - t_0 \text{ and } t \geqslant t_0\},$$

and, in particular, $(\partial u/\partial y)(x, 0, t)$ is supported in

$$\{(x, t)/|x| \leqslant R + t - t_0 \text{ and } t \geqslant t_0\}.$$

Let us denote

$$(A\varphi)(x, t) = -\frac{\partial u}{\partial y}(x, 0, t). \tag{2.6}$$

We shall now extend A to spaces of distributions; unfortunately, we cannot extend A simply by duality, because $\langle A\varphi, \psi \rangle$ equals neither $\langle \varphi, A\psi \rangle$ nor $-\langle \varphi, A\psi \rangle$, for φ and ψ with compact support. On the other hand, we are, indeed, interested mainly by phenomena occurring for finite times, so that it is convenient to set

$$A_T = A \qquad \text{restricted to } [0, T].$$

Let ${}^t A_T$ be defined by

$$
\begin{aligned}
v_{tt} - \Delta v &= 0 && \text{in } Q_T, \\
v(X, T) = v_t(X, T) &= 0 && \text{in } \Omega, \\
v(x, 0, t) &= \psi(x, t) && \text{on } \Sigma_T,
\end{aligned}
\tag{2.7}
$$

and

$$({}^tA_T\psi)(x,t) = -\frac{\partial v}{\partial y}(x,0,t)|_{[0,T]}.$$

Then, if ψ is supported in $B(0,R)\times[0,t_1]$, ${}^tA_T\psi$ is supported in

$$\{(x,t)/|x|\leqslant R+t_1-t \text{ and } t\leqslant t_1\}.$$

We can now check that

$$\langle A_T\varphi,\psi\rangle = \langle\varphi, {}^tA_T\psi\rangle, \tag{2.8}$$

for φ in $\mathscr{D}(\mathbb{R}^n\times(0,T])$ and ψ in $\mathscr{D}(\mathbb{R}^n\times[0,T))$. We have indeed

$$\begin{aligned}
&\langle A_T\varphi,\psi\rangle - \langle\varphi,{}^tA_T\psi\rangle\\
&\quad=\int_0^T\int_\Gamma\left(-\frac{\partial u}{\partial y}(x,0,t)\,v(x,0,t)+\frac{\partial v}{\partial y}(x,0,t)\,u(x,0,t)\right)dx\,dy\\
&\quad=\int_0^T\int_\Omega(\Delta u(x,y,t)\,v(x,y,t)-\Delta v(x,y,t)\,u(x,y,t))\,dx\,dy\\
&\quad=\int_\Omega\left(\int_0^T(u_{tt}v-v_{tt}u)\,dt\right)dx\,dy=0,
\end{aligned}$$

thanks to the initial condition for u and to the final condition for v.

Now, if S belongs to $\mathscr{D}'(\mathbb{R}^n\times[0,T))$, let us define A_TS by

$$\langle A_TS,\varphi\rangle = \langle S,{}^tA_T\varphi\rangle, \qquad \forall\varphi\in\mathscr{D}(\mathbb{R}^n\times(0,T)). \tag{2.9}$$

This relation makes sense because if φ belongs to $\mathscr{D}(\mathbb{R}^n\times[0,T))$, then so does ${}^tA_T\varphi$.

One must remark that, except in the case $N=1$, where the geometry is not rich enough, if φ is supported in a small ball

$$\{(x,t)/|x-x_0|^2+|t-t_0|^2\leqslant r^2\},$$

then $A_T\varphi$ does not vanish in the set

$$\{(x,t)/\exists x_1,t_1: t-t_1=|x-x_1| \text{ and } |x_1-x_0|^2+|t_1-t_0|^2\leqslant r^2\},$$

if φ is suitably chosen. Therefore, geometrically, if S is very singular in a neighborhood of $t=0$, $x=0$, A_TS cannot be small close to the cone $t=|x|$, and we can be in trouble defining (A_TS,φ) is φ is not supported inside $\{(x,t)/t>|x|\}$. One could see this from an analytical point of view by stating that, intuitively, if one develops the symbol of A (see below), it contains powers of τ^{-1}, and therefore involves some integration with respect to t.

Nevertheless, we shall define AS for S in $\mathscr{D}'(\mathbb{R}^n \times [0, \infty))$: let the test function φ be supported in $\mathbb{R}^n \times [0, T]$. We set, for S_T the restriction of S to $[0, T]$,

$$\langle AS, \varphi \rangle = \langle A_T S_T, \varphi \rangle. \tag{2.10}$$

A is well defined: if we have two numbers T and $\hat{T}$, $T < \hat{T}$, for instance, such that φ vanishes outside of $\mathbb{R}^n \times [0, T]$, then ${}^t A_{\hat{T}} \varphi$ vanishes identically on $\mathbb{R}^n \times [T, \hat{T}]$, by the support property of ${}^t A$; therefore,

$$\langle A_T S_T, \varphi \rangle - \langle A_{\hat{T}} S_{\hat{T}}, \varphi \rangle = \langle S_T, {}^t A_T \varphi \rangle - \langle S_{\hat{T}}, {}^t A_{\hat{T}} \varphi \rangle$$

and according to the above remark, this last expression vanishes.

Let

$$H = L^2(\mathbb{R}^n \times (0, T)). \tag{2.11}$$

We define an unbounded operator A_T^0 from H to H by

$$\begin{aligned} D(A_T^0) &= \{\varphi \in H / A_T \varphi \in H\} \\ A_T^0 \varphi &= A_T \varphi, \qquad \forall\, \varphi \in D(A_T^0). \end{aligned} \tag{2.12}$$

In the same fashion, we define an unbounded operator A_T^1 by

$$\begin{aligned} D(A_T^1) = \Big\{ \varphi \in L^2(0, T; H^{1/2}(\mathbb{R}^n)) \Big/ \frac{\partial \varphi}{\partial t} &\in L^2(0, T; H^{-1/2}(\mathbb{R}^n)) \\ \text{and } A_T \varphi &\in L^2(0, T; H^{-1/2}(\mathbb{R}^n)) \Big\} \\ A_T^1 \varphi = A_T \varphi, \qquad \forall\, \varphi \in D(A_T^1). \end{aligned} \tag{2.13}$$

We can now relate problem (2.2)–(2.5) to a problem on the boundary involving the operator A.

THEOREM 0. *Let u_0 belong to $H_0^1(\Omega)$, u_1 to $L^2(\Omega)$ and f to $L^2(Q_T)$. Let w be the solution of*

$$\begin{aligned} \Box w &= f && \textit{in } Q_T, \\ w(X, 0) &= u_0(X) && \textit{on } \Omega, \\ w_t(X, 0) &= u_1(X) && \textit{on } \Omega, \\ w(x, 0, t) &= 0 && \textit{on } \Sigma_T. \end{aligned} \tag{2.14}$$

Define

$$\varphi(x,t)=\frac{\partial w}{\partial y}(x,0,t). \tag{2.15}$$

Then u is a solution of (2.2)–(2.4), *which belongs to* $L^\infty(0,T;H^1(\Omega))\cap W^{1,\infty}(0,T;L^2(\Omega))$ *if and only if*

$$u=w+z \tag{2.16}$$

$$\begin{aligned} \Box z&=0 && \textit{in } Q_T,\\ z(X,0)=z_t(X,0)&=0 && \textit{on } \Omega,\\ z(x,0,t)&=v(x,t) && \textit{on } \Sigma_T, \end{aligned} \tag{2.17}$$

and v belongs to the domain of A_T^1 *and satisfies*

$$\begin{aligned} v&\geqslant 0\\ A_T^1 v&\geqslant 0\\ \langle A_T^1 v-\varphi, v\rangle&=0. \end{aligned} \tag{2.18}$$

Proof. Under the functional assumptions made on u_0, u_1 and f, we know that w is unique and satisfies

$$\int\left(\left|\frac{\partial w}{\partial t}(X,t)\right|^2+|\Delta w(X,t)|^2\right)dx\leqslant c<+\infty, \qquad \forall t\in[0,T].$$

Therefore,

$$\varphi\in L^\infty(0,T;H^{-1/2}(\mathbb{R}^n))\subset L^2(0,T;H^{-1/2}(\mathbb{R}^n))$$

and let, for u in $L^\infty(0,T;H^1(\Omega))\cap W^{1,\infty}(0,T;L^2(\Omega))$,

$$z=u-w.$$

Then, clearly, z satisfies (2.17), so that, in particular,

$$v=z|_{\Sigma_T}\in L^2(0,T;H^{1/2}(\mathbb{R}^n))$$

$$\left.\frac{\partial z}{\partial y}\right|_{\Sigma_T} \quad\text{and}\quad \left.\frac{\partial y}{\partial t}\right|_{\Sigma_T}\in L^2(0,T;H^{-1/2}(\mathbb{R}^n)).$$

Therefore $A_T^1 v$ is well defined; if u is a solution of (2.2)–(2.5), then

$$v=u-w|_{\Sigma_T}\geqslant 0$$

$$A_T^1 v=-\frac{\partial u}{\partial y}(x,0,t)+\frac{\partial w}{\partial y}(x,0,t)\geqslant\varphi$$

and

$$\langle A_T^1 v - \varphi, v \rangle = 0.$$

Conversely, if v satisfies (2.18), clearly $u = w + z$ is a solution of (2.2)–(2.5). ∎

Notice that is we define a multivalued monotone operator β on $\mathbb{R}$ by

$$\beta(r) = \begin{cases} \varnothing & \text{if} \quad r < 0, \\ \mathbb{R} & \text{if} \quad r = 0, \\ \{0\} & \text{if} \quad r > 0 \end{cases}$$

(see [1] for some information on multivalued monotone operators), problem (2.18) is equivalent to

$$\begin{aligned} v &\in D(A_T^1) \\ A_T^1 v + \beta(v) &\ni \varphi. \end{aligned} \tag{2.19}$$

3. The Properties of A, A_T, A_T^0 and A_T^1

3.1. *Representation of A in Fourier Variables*

In what follows, we denote by $\mathscr{F}$ the Fourier transform in x and t, with dual variables ξ and τ; we shall write

$$\begin{aligned} \hat{u}(\xi, y, \tau) &= (\mathscr{F} u)(\xi, y, \tau) \\ \hat{\varphi}(\xi, \tau) &= (\mathscr{F} \varphi)(\xi, \tau). \end{aligned}$$

If we perform a Fourier transform on (2.5), we shall have

$$\begin{aligned} (-\tau^2 + |\xi|^2)\hat{u} - \hat{u}_{yy} &= 0 \\ \hat{u}(\xi, \tau) &= \hat{\varphi}(\xi, \tau), \end{aligned} \tag{3.1}$$

and we wish to obtain an expression of $\hat{u}$ which will give back the unique solution of (2.5) after an inverse Fourier transform $\bar{\mathscr{F}}$.

In the region $|\xi|^2 > \tau^2$, the solution of (3.1) must be

$$\hat{u}(\xi, y, \tau) = \hat{\varphi}(\xi, \tau) \exp(-y \sqrt{|\xi|^2 - \tau^2}) \tag{3.2}$$

so as to avoid infinite energies.

In the regions $|\xi|^2 < \tau^2$, the solution of (3.1) is of the form

$$\hat{u}(\xi, y, \tau) = \hat{\varphi}(\xi, \tau) \cos\left(y\sqrt{\tau^2 - |\xi|^2}\right) + \hat{\psi}(\xi, \tau) \sin\left(y\sqrt{\tau^2 - |\xi|^2}\right),$$

where ψ is some function of φ.

We make use of the Paley–Wiener theorem to determine ψ; we know that if $\varphi = 0$ for $t \leqslant 0$, then we can extend $\varphi(\xi, \cdot)$ as an analytic function of $\tau + ia$, for $a < 0$. On the other hand, $u(x, y, t)$ must be identically zero for $t \leqslant 0$, by the propagation property of A; therefore, $\hat{u}(\xi, y, \cdot)$ must be extended as an analytic function of $\tau + ia$, for $a < 0$. But, this analytic extension is determined by expression (3.2), which gives

$$\hat{u}(\xi, y, \tau + ia) = \hat{\varphi}(\xi, \tau + ia) \exp\left(-y\sqrt{|\xi|^2 - (\tau + ia)^2}\right),$$

where the determination of $\sqrt{|\xi|^2 - (\tau + ia)^2}$ is chosen by continuity at $a = 0^-$ (Fig. 1).

Thus, we shall define the expression

$$\sqrt{|\xi|^2 - \tau^2} = \begin{cases} i\sqrt{\tau^2 - |\xi|^2} & \text{if} \quad \tau > |\xi|, \\ \sqrt{|\xi|^2 - \tau^2} & \text{if} \quad -|\xi| < \tau < |\xi|, \\ -i\sqrt{\tau^2 - |\xi|^2} & \text{if} \quad \tau > -|\xi|, \end{cases} \tag{3.3}$$

and we shall have, for the unique solution u of (3.1),

$$\hat{u}(\xi, y, \tau) = \hat{\varphi}(\xi, \tau) \exp\left(-y\sqrt{|\xi|^2 - \tau^2}\right), \tag{3.4}$$

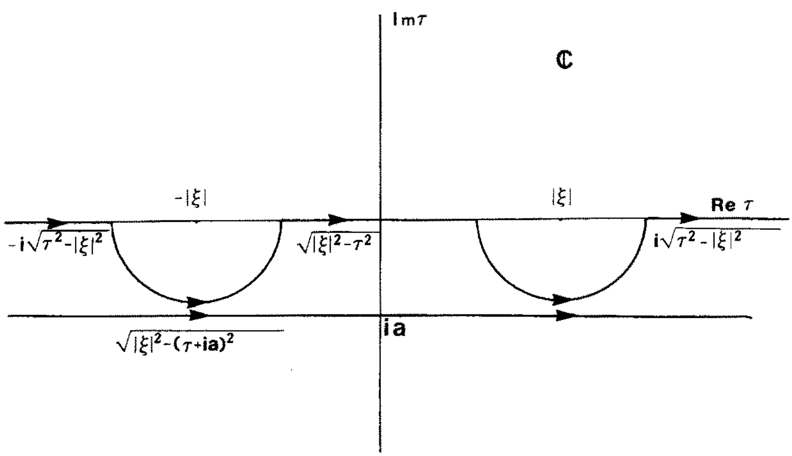

FIG. 1. The representation of the determination of $\sqrt{|\xi|^2 - \tau^2}$ for the symbol of A.

and thus

$$\widehat{A\varphi}(\xi, \tau) = -\left.\frac{\partial u(\xi, y, \tau)}{\partial y}\right|_{y=0} = \hat{\varphi}(\xi, \tau)\sqrt{|\xi|^2 - \tau^2}. \tag{3.5}$$

We remark that the *symbol* of A

$$a(\xi, \tau) = \sqrt{|\xi|^2 - \tau^2} \tag{3.6}$$

does not belong to a reasonable class of pseudo-differential operators, because it is not smooth on an unbounded set, viz., the cone $\tau = \pm |\xi|$. Clearly, A sends continuously $H^s(\mathbb{R}^n \times \mathbb{R})$ to $H^{s-1}(\mathbb{R}^n \times \mathbb{R})$, where s is an arbitrary real number, and H^s the corresponding Sobolev space.

3.2. *First Functional Properties of A_T^0*

The first result wanted here is the following:

THEOREM 1. *Let A_T^0 be defined by* (2.12); *then A_T^0 is injective and $D(A_T^0)$ is included in $C^0([0, T]; L^2(\mathbb{R}^n)) \cap \{u/u(x, 0) = 0$ a.e.$\}$.*

The proof will be done in several steps; the first one will be to define an operator B_a, which will formally be an inverse of A_T^0, and which will be continuous from H to $C^0([0, T]; L^2(\mathbb{R}^n))$.

Let v belong to H and set

$$v_R = v \cdot 1_{\{|x| \leqslant R\}}.$$

In what follows, we shall always identify H with the functions of $L^2(\mathbb{R}^n \times \mathbb{R})$ which vanish outside $\mathbb{R}^n \times [0, T]$.

Define, for $a < 0$,

$$(B_a v_R)(x, t) = \bar{\mathscr{F}}\left(\hat{v}_R(\xi, \tau + ia)\frac{1}{\sqrt{|\xi|^2 - (\tau + ia)^2}}\right), \tag{3.7}$$

where the determination of $\sqrt{|\xi|^2 - (\tau + ia)}$ is obtained by continuity, as in Fig. 1. We can see that $||\xi|^2 - (\tau + ia)^2|$ is bounded away from zero, for all $\tau \in \mathbb{R}$. As $v_R \equiv 0$ for $t \leqslant 0$, it is clear that $\hat{v}_R(\xi, \cdot)$ can be extended to be an analytic function of $\tau + ia$, $a < 0$. Moreover, $v_R(\xi, \tau + ia)$ is continuous in ξ, and

$$\hat{v}_R(\xi, \tau + ia) = \{\mathscr{F}(v_R(x, t)\, e^{at})\}(\xi, \tau) \tag{3.8}$$

which shows that $\hat{v}_R$ is square integrable. Therefore $\tau \to \hat{v}_R(\xi, \tau + ia)/\sqrt{|\xi|^2 - (\tau + ia)^2}$ is square integrable and (3.7) makes sense.

We show now

LEMMA 2. *The operator B_a does not depend on $a < 0$.*

Proof. As $\hat{v}_R(\cdot, \cdot + ia)$ is square integrable, thanks to (3.8), the function

$$\tau \to \hat{v}_R(\xi, \cdot + ia)$$

is square integrable for almost every ξ. But, on the other hand, for every multi-index $p = (p_1, \ldots, p_n)$,

$$\left(\int |D_\xi^p \hat{v}_R(\xi, s - ia)|^2 \, d\xi \, ds \right)^{1/2} \leqslant R^p |v_R|_H,$$

which can be written

$$\hat{v}_R(\cdot, \cdot - ia) \in H^m(\mathbb{R}_\xi^n, L^2(\mathbb{R})), \qquad \forall m \in \mathbb{N}.$$

Therefore, $\hat{v}_R(\cdot, \cdot - ia)$ can be identified with a C^∞ function from $\mathbb{R}_\xi^n$ to $L^2(\mathbb{R})$ and we even have, for every nonpositive a,

for all $p = (p_1, \ldots, p_n)$ and all q, the functions $\xi \to D_\xi^p D_\tau^q \hat{v}_R(\xi, \cdot - ia)$ are continuous and bounded from $\mathbb{R}^n$ to $L^2(\mathbb{R})$. (3.9)

Define now a function h_a by

$$h_a(\xi, t) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{it(\tau + ia)} \frac{1}{\sqrt{|\xi|^2 - (\tau + ia)^2}} \hat{v}_R(\xi, t + ia) \, d\tau. \tag{3.10}$$

According to the previous considerations and the fact that

$$e^{it(\tau + ia)} = e^{it\tau} e^{-at},$$

and

$$\left| \frac{1}{\sqrt{|\xi|^2 - (\tau + ia)^2}} \right| \leqslant \frac{c}{|\tau|} \qquad \text{for large } |\tau|,$$

we deduce that h_a is well defined.

Moreover, we have the estimate

$$|h_a(\xi, \tau)| \leqslant \frac{1}{2\pi} \left(\int_{\mathbb{R}} |\hat{v}_R(\xi, \tau + ia)|^2 \, d\tau \right)^{1/2} e^{-at}$$
$$\times \left(\int_{\mathbb{R}} \frac{d\tau}{||\xi|^2 - (\tau + ia)^2|} \right)^{1/2}.$$

Let now a and b be two negative numbers. Consider the contour integral

$$\int_{\gamma_r} \psi(z)\, dz$$

with

$$\psi(z) = e^{itz} \frac{1}{\sqrt{|\xi|^2 - z^2}} \hat{v}_R(\xi, z),$$

and γ_r depicted in Fig. 2.

Clearly, ψ is holomorphic in $\operatorname{Im} z < 0$; we have

$$\begin{aligned} \left| \int_b^a \psi(r + i\rho)\, d\rho \right| &\leqslant \frac{1}{|a|} \int_{-a}^{-b} e^{t\rho}\, |\hat{v}_R(\xi, r + i\rho)|\, d\rho \\ &\leqslant \frac{e^{t|b|}}{|a|} \int_{-a}^{-b} |\hat{v}_R(\xi, \tau + i\rho)|\, d\rho. \end{aligned}$$

The second derivative in τ of $\hat{v}_R$, $(\partial^2 \hat{v}_R / \partial \tau^2)(\xi, \cdot + ia)$, satisfies, thanks to (3.9),

$$\int \left| \frac{\partial^2 \hat{v}_R}{\partial \tau^2} (\xi, \tau + ia) \right|^2 d\tau \leqslant c < \infty, \qquad \forall \xi.$$

Thus, $\hat{v}_R(\xi, r + ia)$ tends to zero when $|r|$ tends to infinity. Moreover, $\hat{v}_R(\xi, \tau + ia)$ is uniformly bounded in ξ, in τ and in $a > 0$. The Lebesgue convergence theorem implies that

$$\lim_{|r| \to \infty} \int_{-a}^{-b} |\hat{v}_R(\xi, r + i\rho)|\, d\rho = 0,$$

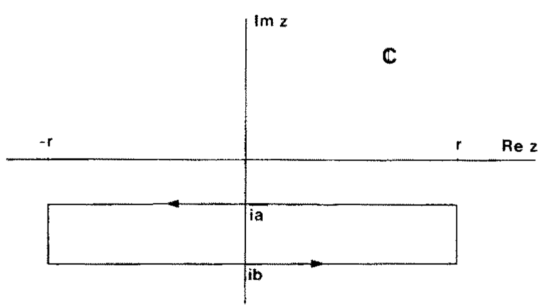

FIG. 2. The integration contour γ_r.

and therefore, thanks to Cauchy's theorem,

$$\int_{\mathbb{R}} \psi(\tau + ia)\, d\tau = \int_{\mathbb{R}} \psi(\tau + ib)\, d\tau.$$

We shall now denote $h(\xi, t)$ instead of $h_a(\xi, t)$; $B_a v_R$ is the inverse, partial in ξ, Fourier transform of $h(\xi, t)$; we can see now that B_a does not depend on a; from this moment, B_a will be denoted B. ∎

We shall now estimate h in terms of t and ξ.

PROPOSITION 3. *Let*

$$\theta(\xi) = \left(\int |\hat{v}_R(\xi, \tau)|^2\, d\tau\right)^{1/2}. \tag{3.11}$$

Then, there exist constants C_1 and C_2 such that, for all $a < 0$,

$$|h(t, \xi)| \leqslant e^{-at} C_1 (a^2 + |\xi|^2)^{-1/4} (1 - C_2 \operatorname{Log}(-a/\sqrt{a^2 + |\xi|^2}))^{1/2} \theta(\xi). \tag{3.12}$$

Proof. Denote for $a \leqslant 0$,

$$\theta_a(\xi) = \left(\int |\hat{v}_R(\xi, \tau + ia)|^2\, d\tau\right)^{1/2}.$$

We deduce from (3.9) that θ_a is finite for every nonpositive a. Moreover, we have

$$\theta_a(\xi) \leqslant \theta_0(\xi) \equiv \theta(\xi), \qquad \forall\, \xi. \tag{3.13}$$

Denote by $\mathscr{F}_1$ the partial Fourier transform with respect to x, and by $\mathscr{F}_2$ the partial Fourier transform with respect to t. Then

$$\begin{aligned}
|\theta_a(\xi)|^2 &= \int |\hat{v}_R(\xi, \tau + ia)|^2\, d\tau \\
&= \int |\bar{\mathscr{F}}_2(\hat{v}_R(\xi, \cdot + ia))(t)|^2\, dt \\
&= \int |\bar{\mathscr{F}}_2 \mathscr{F}(v_R(\cdot, \cdot)\, e^{at})|^2\, dt \\
&= \int |[\mathscr{F}_1(v_R(\cdot, t))](\xi)|^2\, e^{2at}\, dt \\
&\leqslant \int |[\mathscr{F}_1(v_R(\cdot, t))](\xi)|^2\, dt \\
&= |\theta(\xi)|^2,
\end{aligned}$$

which proves (3.13).

We apply the Cauchy–Schwartz inequality to the integral which defines h, (3.10):

$$|h(t,\xi)| \leqslant |\hat{v}_R(\xi,\cdot+ia)|_{L^2(\mathbb{R})}\, e^{-at} \left(\int_{\mathbb{R}} \frac{d\tau}{|(\tau+ia)^2-|\xi|^2|}\right)^{1/2}$$

so that

$$|h(t,\xi)| \leqslant \theta(\xi) \left(\int_{\mathbb{R}} \frac{d\tau}{|(\tau+ia)^2-|\xi|^2|}\right)^{1/2} e^{-at}. \tag{3.14}$$

Thus, we have to evaluate

$$I = \left(\int_{-\infty}^{+\infty} ((\tau^2-a^2-|\xi|^2)^2+4a^2\tau^2)^{-1/2}\, d\tau\right)^{1/2}.$$

We perform the change of variable $\tau=\sqrt{a^2+|\xi|^2}\, s$, and we obtain

$$I^2 = (a^2+|\xi|^2)^{-1/2} \left\{\int_{-\infty}^{\infty} [(s^2-1)^2+4(a^2+|\xi|^2)^{-1}s^2a^2]^{-1/2}\, ds\right\},$$

and by symmetry,

$$I^2 = 4(a^2+|\xi|^2)^{-1/2} \left\{\int_0^1 [(s^2-1)^2+4a^2s^2(a^2+|\xi|^2)^{-1}]^{-1/2}\, ds\right\}. \tag{3.15}$$

Let

$$\lambda = \frac{-a}{\sqrt{a^2+|\xi|^2}} \tag{3.16}$$

and

$$s = 1-\lambda y.$$

Then,

$$\begin{aligned}
&\int_0^1 [(s^2-1)^2+4a^2s^2(a^2+|\xi|^2)^{-1}]^{-1/2}\, ds \\
&\quad = \int_0^{1/\lambda} \lambda[4\lambda^2(1-\lambda y)^2+\lambda^2y^2(2-\lambda y)^2]^{-1/2}\, dy \\
&\quad = J.
\end{aligned}$$

This integral is cut into two pieces, one from 0 to C, and the other one from C to $1/\lambda$, where C is, at the present moment, arbitrarily chosen between 0 and $1/\lambda$. Thus

$$J \leqslant \int_0^C \frac{dy}{2(1-\lambda y)} + \int_C^{1/\lambda} \frac{dy}{y(2-\lambda y)}$$
$$\leqslant \int_0^C \frac{dy}{2(1-\lambda y)} + \int_C^{1/\lambda} \frac{dy}{y},$$

so that,

$$J \leqslant -\frac{1}{2\lambda} \operatorname{Log}(1-\lambda C) - \operatorname{Log} \lambda - \operatorname{Log} C.$$

Choose now

$$C = \frac{2}{1+2\lambda};$$

then

$$J \leqslant -\frac{1}{2\lambda} \operatorname{Log} \frac{1}{1+2\lambda} - \operatorname{Log} 2 + \operatorname{Log}(1+2\lambda) - \operatorname{Log} \lambda.$$

In a neighborhood of $\lambda = 0$, the term $(1/(2\lambda)) \operatorname{Log}(1/(1+2\lambda))$ is bounded; (3.16) shows that λ is always between 0 and 1, so that we can find constants C_1 and C_2 such that

$$J \leqslant C_1^2(1 - C_2 \operatorname{Log} \lambda). \tag{3.17}$$

If we make use of (3.17) and (3.16), we obtain

$$I \leqslant 2(a^2 + |\xi|^2)^{-1/4} C_1 (1 - C_2 \operatorname{Log}(-a/\sqrt{a^2 + |\xi|^2}))^{1/2} e^{-at},$$

which yields (3.12), with the help of (3.14), after renaming constant C_1. ∎

The estimate (3.12) enables us to define Bv for an arbitrary v in H; we have the estimate

$$\int |Bv_R(x,t)|^2 \, dx = \int |\bar{\mathscr{F}}_1 h(\cdot, t)(x)|^2 \, dx = \int |h(\xi, t)|^2 \, d\xi,$$

and if we weaken (3.12) to

$$|h(t,\xi)| \leqslant e^{-at} C_3 \theta(\xi),$$

we obtain

$$\int |Bv_R(x,t)|^2\,dx \leqslant e^{-2at} C_3^2 \int |\hat{v}_R(\xi,\tau)|^2\,d\tau\,d\xi,$$

which proves that, for arbitrary R, and T',

$$\int_0^{T'} \int_{\mathbb{R}^n} |Bv_R(x,t)|^2\,dx\,dt \leqslant C(T')\,|v|_H^2.$$

Therefore, we may pass to the limit as R goes infinity.

We may remark $Bv \in L^2_{\mathrm{Loc}}(\mathbb{R}; H^{1/2-\varepsilon}(\mathbb{R}^n))$ for every positive ε. Let us now prove that Bv is continuous from $\mathbb{R}$ to $L^2(\mathbb{R}^n)$.

LEMMA 4. *There exists a continuous, nonnegative function $g(s,t)$ which vanishes for $s=t$, such that*

$$|h(t,\xi) - h(s,\xi)| \leqslant g(s,t)\,\theta(\xi). \tag{3.18}$$

Proof. We resume working with v_R; we have for negative a

$$\begin{aligned}|h(t,\xi)-h(s,\xi)| &= \left| \frac{1}{2\pi} \int_{\operatorname{Im} z=a} \frac{e^{itz}-e^{isz}}{\sqrt{|\xi|^2 - z^2}}\, v_R(\xi,z)\,dz \right| \\ &\leqslant \theta_a(\xi) \left[\int_{\operatorname{Im} z=a} \frac{|e^{itz}-e^{isz}|^2}{||\xi|^2 - z^2|}\,dz \right]^{1/2}.\end{aligned}$$

We shall compute the above integral in the spirit of the proof of Proposition 3. Let

$$I = \int_{\operatorname{Im} z=a} \frac{|e^{itz}-e^{isz}|^2}{||\xi|^2 - z^2|}\,dz.$$

Then

$$\begin{aligned} I &= \int_{\operatorname{Im} z=a} \frac{|e^{itz}-e^{isz}|^2}{1+|z|}\,\frac{1+|z|}{||\xi|^2 - z^2|}\,dz \\ &\leqslant \left(\int_{\operatorname{Im} z=a} \frac{|e^{itz}-e^{isz}|^4}{(1+|z|)^2}\,dz \right)^{1/2} \left(\int \frac{(1+|z|)^2}{||\xi|^2 - z^2|^2}\,dz \right)^{1/2}. \end{aligned}$$

Set

$$g(s,t,a) = \left(\int_{\operatorname{Im} z=a} \frac{|e^{isz}-e^{itz}|^4}{(1+|z|)^2}\,dz \right)^{1/2}. \tag{3.19}$$

The theorem of Lebesgue tells us that g is continuous with respect to s and t and vanishes for $s = t$. Thus, we have to evaluate

$$\int_{\operatorname{Im} z = a} \frac{(1+|z|)^2}{||\xi|^2 - z^2|^2}\, dz = \int_{-\infty}^{\infty} \frac{(1+|\sigma + ia|)^2}{(\sigma^2 - a^2 - |\xi|^2)^2 + 4a^2\sigma^2}\, d\sigma \tag{3.20}$$

and to show that (3.20) is bounded inedependently of $|\xi|$.

We estimate $(1+|\sigma+ia|)^2$ by $C(1+\sigma^2)$; we notice that

$$\begin{aligned}\int_0^{+\infty} \frac{d\sigma}{(\sigma^2 - a^2 - |\xi|^2)^2 + 4a^2\sigma^2} &\leqslant \int_0^{\sqrt{(a^2+|\xi|^2)/2}} \frac{d\sigma}{((a^2+|\xi|^2)/2)^2} \\ &\quad + \int_{\sqrt{(a^2+|\xi|^2)/2}}^{+\infty} \frac{d\sigma}{4a^2\sigma^2} \\ &\leqslant C((a^2+|\xi|^2)^{-3/2} + (a^2+|\xi|^2)^{-1/2})\end{aligned}$$

which gives a uniform bound in ξ. Therefore, it remains to estimate uniformly in $|\xi|$

$$J = \int_{-\infty}^{\infty} \frac{\sigma^2}{(\sigma^2 - a^2 - |\xi|^2)^2 + 4a^2\sigma^2}\, d\sigma = 2\int_0^{\infty} \frac{\sigma^2}{(\sigma^2 - a^2 - |\xi|^2)^2 + 4a^2\sigma^2}\, d\sigma. \tag{3.21}$$

We cut J into two pieces, one from 0 to $2\sqrt{a^2+|\xi|^2}$, and the other one from $2\sqrt{a^2+|\xi|^2}$ to $+\infty$. If

$$\sigma \geqslant 2\sqrt{a^2+|\xi|^2}$$

then

$$\sigma^2 - (a^2+|\xi|^2) \geqslant 3\sigma^2/4,$$

and thus

$$\begin{aligned}\int_{2(a^2+|\xi|^2)^{1/2}}^{+\infty} \frac{\sigma^2\, d\sigma}{(\sigma^2 + a^2 - |\xi|^2)^2 + 4a^2\sigma^2} &\leqslant \int_{2(a^2+|\xi|^2)^{1/2}}^{+\infty} \frac{\sigma^2\, d\sigma}{(9\sigma^4/16)} \\ &= \frac{8}{9(a^2+|\xi|^2)^{1/2}}.\end{aligned} \tag{3.22}$$

Let us estimate the other part of J,

$$\int_0^{2(a^2+|\xi|^2)^{1/2}} \frac{\sigma^2\, d\sigma}{(\sigma^2 - a^2 - |\xi|^2)^2 + 4a^2\sigma^2},$$

which becomes, after performing the change of variables $\sigma^2 = s(a^2 + |\xi|^2)$,

$$\frac{1}{2}(a^2 + |\xi|^2)^{-1/2} \int_0^{\sqrt{2}} \frac{\sqrt{s}\, ds}{(s-1)^2 + 4a^2 s(a^2 + |\xi|^2)^{-1}}. \tag{3.23}$$

Set

$$\lambda = \frac{2a^2}{a^2 + |\xi|^2};$$

then

$$K = \int_0^{\sqrt{2}} \frac{\sqrt{s}\, ds}{(s-1)^2 + 4a^2 s(a^2 + |\xi|^2)^{-1}} \leqslant 2^{1/4} \int_0^{\sqrt{2}} \frac{ds}{(s-1)^2 + 2\lambda s}.$$

If we take a new variable $s - 1 = s'$, we estimate now

$$\int_{-1}^{\sqrt{2}-1} \frac{ds'}{s'^2 + 2\lambda(s'+1)}.$$

But this last expression can be explicitly integrated, because

$$\begin{aligned}
&\int_{-1}^{\sqrt{2}-1} \frac{ds'}{s'^2 + 2\lambda(s'+1)} \\
&\quad = \int_{-1}^{\sqrt{2}-1} \frac{ds'/\sqrt{2\lambda - \lambda^2}}{(2\lambda - \lambda^2)[((s'+\lambda)^2/2\lambda - \lambda^2) + 1]} \sqrt{2\lambda - \lambda^2} \\
&\quad = \frac{1}{\sqrt{2\lambda - \lambda^2}} \left\{ \operatorname{arc\,tg} \frac{\sqrt{2} - 1 + \lambda}{\sqrt{2\lambda - \lambda^2}} - \operatorname{arc\,tg} \frac{\lambda - 1}{\sqrt{2\lambda - \lambda^2}} \right\} \\
&\quad \leqslant \frac{C'}{\sqrt{\lambda}}.
\end{aligned}$$

Finally,

$$\begin{aligned}
&\int_0^{2\sqrt{a^2 + |\xi|^2}} \frac{\sigma^2\, d\sigma}{(\sigma^2 - a^2 - |\xi|^2)^2 + 4a^2\sigma^2} \\
&\qquad \leqslant \frac{C' \sqrt{a^2 + |\xi|^2}}{a\sqrt{2}} \frac{1}{2} \sqrt{a^2 + |\xi|^2} = \frac{C''}{a},
\end{aligned}$$

which gives an estimate on J independently of $|\xi|$. Therefore, (3.20) is bounded independently of $|\xi|$, which proves Lemma 4. ∎

End of the proof of Theorem 1. We deduce from Lemmas 3 and 4 that B is linear, continuous from H to $C^0(\mathbb{R}; L^2(\mathbb{R}^n)) \cap L^2_{\mathrm{Loc}}(\mathbb{R}; H^{1/2-\varepsilon}(\mathbb{R}^n))$, for all positive ε. Moreover, if v belongs to $H \cap \mathscr{S}_{t \geqslant 0}$, the very definition of B (see (3.7)) shows that Bv is supported in $\mathbb{R}^n \times \mathbb{R}^+$, thanks to the Paley–Wiener theorem. In particular, for such a v,

$$Bv(\cdot, t) = 0 \qquad \text{for} \quad t \leqslant 0.$$

As $H \cap \mathscr{S}_{t \geqslant 0}$ is dense in H, we can see that for every v in H,

$$Bv(\cdot, t) = 0, \qquad \forall\, t \leqslant 0, \forall\, v \in H. \tag{3.24}$$

Define

$$B_T v = (Bv)\, 1_{[0,T]}(t). \tag{3.25}$$

Then B_T is linear continuous from H to $C^0([0, T]; L^2(\mathbb{R}^n)) \cap \{u/u(\cdot, 0) = 0\}$.

From its definition, B is injective, and so is B_T. On the other hand, if we set for a negative and v in $\mathscr{S}_{t \geqslant 0}$,

$$A_a(v) = \bar{\mathscr{F}}(\hat{v}(\xi, \tau + ia) \sqrt{|\xi|^2 - (\tau + ia)^2}),$$

we can show, in the fashion of Lemma 2, that A_a does not depend on A, and that, in fact, $A = A_a$, thanks to (3.5). Therefore,

$$ABv = v, \qquad \forall\, v \in \mathscr{S}_{t \geqslant 0}.$$

By density

$$ABv = v, \qquad \forall\, v \in H. \tag{3.26}$$

Let us show now that

$$(AB_T v)\, 1_{[0,T]} = v, \qquad \forall\, v \in H. \tag{3.27}$$

We have only to see that $B_T v - Bv$ is supported in $\mathbb{R}^n \times [0, \infty)$, and therefore $A(B_T v - Bv)$ is supported in $\mathbb{R}^n \times [0, \infty)$ too. From (3.27), B_T is injective. If u belongs to $D(A_T^0)$ defined by (2.12), we know that

$$A_T^0 u = Au|_{[0,T]};$$

therefore, (3.27) can be rewritten as

$$A_T^0 B_T v = v, \qquad \forall\, v \in H. \tag{3.28}$$

Conversely, if $u \in D(A_T^0)$, then $A_T^0 u \in H$, and $B_T A_T^0 u$ is well defined. It follows that

$$B_T A_T^0 u = B(Au)\, 1_{[0,T]},$$

and knowing that $B(Au) = u$, we can see that

$$B_T A_T^0 u = u, \qquad \forall u \in D(A_T^0). \tag{3.29}$$

Relations (3.28) and (3.29) show that B_T can be identified with the inverse of A_T^0, and this proves Theorem 1. ∎

COROLLARY 5. *A_T^0 is closed in H.*

Proof. This is immediate because $(A_T^0)^{-1}$ is bounded and therefore closed. ∎

3.3. *Positivity Properties of A_T*

The main result is

THEOREM 6. *Operators A_T^0 and $(A_T^0)^*$ are positive; moreover, for u in $D(A_T^0)$, the integral*

$$\int_{|\tau| \leqslant |\xi|} \hat{u}(\tau, \xi) \sqrt{|\xi|^2 - \tau^2}\, d\xi\, d\tau \tag{3.30}$$

is defined, and

$$(A_T^0 u, u) = (2\pi)^{-n-1} \int_{|\tau| \leqslant |\xi|} |\hat{u}(\tau, \xi)|^2 \sqrt{|\xi|^2 - \tau^2}\, d\xi\, d\tau$$
$$+ \tfrac{1}{2} \int |u(x, T)|^2\, dx. \tag{3.31}$$

Proof. Assume first that u belongs to $\mathscr{S}(\mathbb{R}^n \times (0, \infty))$. Let us show then that $u_T = u \cdot 1_{[0,T]}$ belongs to the domain of A_T. The function Au belongs indeed to $\mathscr{S}$, and, in particular, $v = Au \cdot 1_{[0,T]}$ belongs to H. Then

$$B_T v = B(Au \cdot 1_{[0,T]})\, 1_{[0,T]},$$

and by the support property of B and A,

$$B_T v = u \cdot 1_{[0,T]},$$

which proves our contention, and that

$$A_T u = Au \cdot 1_{[0,T]}.$$

Set

$$\tilde{u}(\xi, t) = \int u(x, t)\, e^{-ix\xi}\, dx$$

$$\tilde{v}(\xi, t) = \int v(x, t)\, e^{-ix\xi}\, dx.$$

Then

$$(A_T^0 u, u) = \int_0^T dt \int_{\mathbb{R}^n} [A_T^0 u(x, t)]\, u(x, t)\, dx$$

$$= \mathrm{Re} \int_0^T dt \int_{\mathbb{R}^n} \tilde{v}(\xi, t)\, \bar{\tilde{u}}(\xi, t)\, d\xi (2\pi)^{-n}.$$

Thus, we have to evaluate

$$\int_{\mathbb{R}^n} \tilde{v}(\xi, t)\, \bar{\tilde{u}}(\xi, t)\, d\xi (2\pi)^{-n}. \tag{3.32}$$

We notice that

$$\tilde{u}(\xi, t) = \frac{1}{2\pi} \int e^{i\tau t} \hat{u}(\tau, \xi)\, d\tau \tag{3.33}$$

$$\tilde{v}(\xi, t) = \frac{1}{2\pi} \int e^{i\tau t} \sqrt{|\xi|^2 - \tau^2}\, \hat{u}(\tau, \xi)\, d\tau. \tag{3.34}$$

For ξ and T fixed, we define a pseudo-differential operator $\mathscr{A}_\xi$ by

$$(\mathscr{A}_\xi w)(t) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\tau t} \sqrt{|\xi|^2 - \tau^2}\, w(\tau)\, d\tau, \qquad w \in \mathscr{S},$$

where $\sqrt{|\xi| - \tau^2}$ is the determination defined in (3.3). Let A_ξ be defined by

$$D(A_\xi) = \{w \in L^2(0, T)/(\mathscr{A}w) \cdot 1_{[0,T]} \in L^2(0, T)\}$$
$$A_\xi w = (\mathscr{A}_\xi w) \cdot 1_{[0,T]}.$$

We can decompose $\mathscr{A}_\xi$ as

$$\mathscr{A}_\xi = \frac{d}{dt} + \mathscr{C}_\xi, \tag{3.35}$$

where the symbol of $\mathscr{C}_\xi$ is

$$c(\xi, \tau) = \sqrt{|\xi|^2 - \tau^2} - i\tau.$$

For $\tau > |\xi|$,

$$c(\xi, \tau) = i(\sqrt{\tau^2 - |\xi|^2} - \tau),$$

for $\tau < -|\xi|$,

$$c(\xi, \tau) = i(\sqrt{\tau^2 - |\xi|^2} + \tau),$$

so that c is bounded.

Define an operator C_ξ by

$$D(C_\xi) = \{w \in L^2(0, T)/(\mathscr{C}_\xi w) \cdot 1_{[0,T]} \in L^2(0, T)\}$$
$$C_\xi w = (\mathscr{C}_\xi w) \cdot 1_{[0,T]}.$$

Clearly, $D(C_\xi) = L^2(0, T)$ and C_ξ is bounded.

Arguing as in the proof of Theorem 1, we set, for $a < 0$,

$$(\mathscr{B}_\xi w)(t) = \frac{1}{2\pi} \int_{\operatorname{Im} \tau = a} e^{i\tau t} (\sqrt{|\xi|^2 - \tau^2})^{-1} w(\tau)\, d\tau,$$

which is independent from a (same proof as Lemma 2), and moreover, we have the estimate

$$|\mathscr{B}_\xi w|_{H^1(0,\infty)} \leqslant c\, |w|_{L^2(0,\infty)}, \tag{3.36}$$

which comes from the fact that the symbol $b(\xi, \tau) = (\sqrt{|\xi|^2 - \tau^2})^{-1}$ satisfies the estimate

$$|b(\xi, \tau)|^{-1} \geqslant \max(a, |\tau| - |\xi|),$$

so that

$$\begin{aligned} |b(\xi, \tau)| &\leqslant a^{-1} && \text{for} \quad |\tau| \leqslant |\xi| + a, \\ |b(\xi, \tau)| &\leqslant (|\tau| - |\xi|)^{-1} && \text{for} \quad |\tau| \geqslant |\xi| + a. \end{aligned} \tag{3.37}$$

From (3.37), we can see that (3.36) holds. In particular, $B_\xi w$ is continuous because $H^1(0, \infty)$ is a set of continuous functions. Arguing still as in the proof of Theorem 1, we observe that $B_\xi = 1_{[0,T]} \cdot \mathscr{B}_\xi$ is the inverse of A_ξ, and therefore

$$D(A_\xi) \subset \{u \in C^0([0, T])/u(0) = 0\};$$

thanks to the decomposition

$$A_\xi = \frac{d}{dt} + C_\xi,$$

we can see that

$$D(A_\xi) = \{u \in H^1([0, T])/u(0) = 0\}. \tag{3.38}$$

Therefore, if w is infinitely differentiable and vanishes for $t \leqslant 0$, then $w_T = w \cdot 1_{[0,T]}$ belongs to $D(A_\xi)$.

Let

$$w_\varepsilon = \begin{cases} w & \text{if} \quad t \leqslant T, \\ w(T)(T+\varepsilon-t)\,\varepsilon^{-1} & \text{if} \quad T \leqslant t \leqslant T+\varepsilon, \\ 0 & \text{if} \quad t \geqslant T+\varepsilon, \end{cases}$$

and let us estimate $\operatorname{Re}(A_\xi w_\varepsilon, w_\varepsilon)$:

$$\begin{aligned} \operatorname{Re}(A_\xi w_\varepsilon, w_\varepsilon) &= \operatorname{Re}\int_0^T A_\xi w_\varepsilon \bar{w}_\varepsilon\, dt \\ &= \operatorname{Re}\int_{\mathbb{R}} \mathscr{A}_\xi w_\varepsilon \bar{w}_\varepsilon\, dt - \operatorname{Re}\int_T^{T+\varepsilon} \mathscr{A}_\xi w_\varepsilon \bar{w}_\varepsilon\, dt. \end{aligned}$$

We observe that

$$\int_{\mathbb{R}} (\mathscr{A}_\xi w_\varepsilon)\, \bar{w}_\varepsilon\, dt = \frac{1}{2\pi}\int_{\mathbb{R}} \sqrt{|\xi|^2-\tau^2}\, |\hat{w}_\varepsilon(\tau)|^2\, d\tau,$$

and, in particular,

$$\operatorname{Re}\int_{\mathbb{R}} (\mathscr{A}_\xi w_\varepsilon)\, \bar{w}_\varepsilon\, dt = \frac{1}{2\pi}\int_{|\xi|\geqslant|\tau|} \sqrt{|\xi|^2-\tau^2}\, |\hat{w}_\varepsilon(\tau)|^2\, d\tau.$$

Let us prove that, when ε tends to zero,

$$\int_{|\xi|\geqslant|\tau|} \sqrt{|\xi|^2-\tau^2}\, |\hat{w}_\varepsilon(\tau)|^2\, d\tau \to \int_{|\xi|\geqslant|\tau|} \sqrt{|\xi|^2-\tau^2}\, |\hat{w}_T(\tau)|^2\, d\tau. \tag{3.39}$$

This is of course equivalent to

$$\lim_{\varepsilon\to 0}\int_{|\xi|\geqslant|\tau|} \sqrt{|\xi|^2-\tau^2}\, |\hat{w}_\varepsilon(\tau)-\hat{w}_T(\tau)|^2\, d\tau = 0. \tag{3.40}$$

But

$$(\hat{w}_T-\hat{w}_\varepsilon)(\tau) = -e^{iT\tau}(e^{-i\varepsilon\tau}-1+\varepsilon i\tau)(\varepsilon\tau^2)^{-1}w(T).$$

We can see that $\hat{w}_T-\hat{w}_\varepsilon$ converges uniformly to zero on $[-|\xi|, |\xi|]$, which proves (3.40), and therefore (3.41).

On the other hand,

$$\int_T^{T+\varepsilon} (\mathscr{A}_\xi w_\varepsilon)\, \bar{w}_\varepsilon\, dt = \int_T^{T+\varepsilon} \frac{dw_\varepsilon}{dt}\, \bar{w}_\varepsilon\, dt + \int_T^{T+\varepsilon} (\mathscr{C}_\xi w_\varepsilon)\, \bar{w}_\varepsilon\, dt. \tag{3.41}$$

We notice that $\mathscr{C}_\xi w_\xi$ is bounded in $L^2(\mathbb{R})$ and that $\bar{w}_\varepsilon 1_{[T,T+\varepsilon]}$ converges to zero in $L^2(\mathbb{R})$; therefore, $\int_T^{T+\varepsilon}(\mathscr{C}_\xi w_\varepsilon)\, \bar{w}_\varepsilon\, dt$ converges to zero, as ε tends to zero.

The term $\int_T^{T+\varepsilon}(dw_\varepsilon/dt)\, \bar{w}_\varepsilon\, dt$ equals $-\frac{1}{2}|w(T)|^2$. Finally,

$$\operatorname{Re}\,(A_\xi w, w) = \frac{1}{2\pi}\int_{|\tau|\leqslant|\xi|} |\hat{w}_T(\tau)|^2 \sqrt{|\xi|^2-\tau^2}\, d\tau + \frac{1}{2}|w(T)|^2. \quad (3.42)$$

From (3.42), we can see that

$$\int_{\mathbb{R}^n} \tilde{v}(\xi, t)\, \bar{\tilde{u}}(\xi, t)\, d\xi (2\pi)^{-n}$$

$$= \frac{1}{2\pi}\int_{|\tau|\leqslant|\xi|} |u_T(\xi, \tau)|^2 \sqrt{|\xi|^2-\tau^2}\, d\tau\, d\xi + \int |u(\xi, T)|^2\, d\xi,$$

and, at last,

$$(A_T^0 u_T, u_T) = (2\pi)^{-n-1}\int_{|\tau|\leqslant|\xi|} |\hat{u}_T(\xi, \tau)|^2 \sqrt{|\xi|^2-\tau^2}\, d\tau\, d\xi$$

$$+ \int |u(x, T)|^2 dx, \quad (3.43)$$

where $u_T = u \cdot 1_{[0,T]}$ and $u \in \mathscr{S}(\mathbb{R}^n \times (0, \infty))$.

Let us show now that (3.43) holds for arbitrary elements u of $D(A_T^0)$. Given u in $D(A_T^0)$, let

$$v = A_T^0 u,$$

and let v^ε be a sequence of elements of $\mathscr{S}(\mathbb{R}^n \times (0, T))$ such that

$$v^\varepsilon \to u \qquad \text{in } H. \quad (3.44)$$

Let α be positive and

$$\begin{aligned} u^\varepsilon &= Bv^\varepsilon \\ v_\alpha^\varepsilon &= \bar{\mathscr{F}}(\hat{v}^\varepsilon/(1+\alpha|\xi|)^{1/2}) \\ v_\alpha &= \bar{\mathscr{F}}(\hat{v}/(1+\alpha|\xi|)^{1/2}) \\ u_\alpha^\varepsilon &= Bv_\alpha^\varepsilon \\ u_\alpha &= Bv_\alpha. \end{aligned}$$

Then, we observe that v_α^ε belongs to $\mathscr{S}(\mathbb{R}^n \times (0, T))$ and that u^ε and u_α^ε belong to $C^0(\mathbb{R}; L^2(\mathbb{R}^n))$, with $u^\varepsilon \cdot 1_{[0,T]}$ and $u_\alpha^\varepsilon \cdot 1_{[0,T]}$ being elements of H.

We know that

$$u^\varepsilon = (2\pi)^{-n+1} \int_{\operatorname{Im}\tau = a} \int_{\mathbb{R}} \frac{\hat{v}^\varepsilon(\xi,\tau)}{\sqrt{|\xi|^2 - \tau^2}} e^{i(t\tau + x\cdot\xi)}\, d\xi\, d\tau$$

$$= (2\pi)^{-n+1} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \left(\int_{\operatorname{Im}\tau = a} e^{it\tau} \frac{\hat{v}^\varepsilon(\xi,\tau)}{\sqrt{|\xi|^2 - \tau^2}} e^{it\tau} \frac{\hat{v}^\varepsilon(\xi,\tau)}{\sqrt{|\xi|^2 - \tau^2}}\, d\tau \right) d\xi,$$

and that, in the same fashion,

$$u_\alpha^\varepsilon = (2\pi)^{-n-1} \int_{\mathbb{R}^n} \frac{e^{ix\cdot\xi}}{(1+\alpha|\xi|)} \left(\int_{\operatorname{Im}\tau = a} \frac{\hat{v}^\varepsilon(\xi,\tau)\, e^{it\tau}}{\sqrt{|\xi|^2 - \tau^2}}\, d\tau \right) d\xi,$$

so that

$$\hat{u}_\alpha^\varepsilon = \hat{u}^\varepsilon (1+\alpha|\xi|)^{-1/2}, \tag{3.45}$$

and denoting $\hat{\psi}_T(\tau) = \int_0^T e^{-it\tau}\, dt$, we obtain

$$\begin{aligned} \mathscr{F}(u_\alpha^\varepsilon \cdot 1_{[0,T]}) &= \hat{\psi}_T(\tau) * \hat{u}_\alpha^\varepsilon \\ &= \hat{\psi}_T(\tau) * [\hat{u}^\varepsilon (1+\alpha|\xi|)^{-1/2}] \\ &= [\hat{\psi}_T * \hat{u}^\varepsilon](1+\alpha|\xi|)^{-1/2} \\ &= \mathscr{F}(u^\varepsilon \cdot 1_{[0,T]})(1+\alpha|\xi|)^{-1/2}. \end{aligned}$$

Thus, we deduce that $u^\varepsilon \cdot 1_{[0,T]}$ and $u_\alpha^\varepsilon \cdot 1_{[0,T]}$ belong to the domain of A_T^0. Moreover,

$$\begin{aligned} &\int |\xi|\, |(u_\alpha^\varepsilon \cdot 1_{[0,T]} - u_\alpha \cdot 1_{[0,T]})^\wedge|^2\, d\xi\, d\tau \\ &\quad = \int |\xi|\, (1+\alpha|\xi|)^{-1} |(u^\varepsilon \cdot 1_{[0,T]} - u \cdot 1_{[0,T]})^\wedge|^2\, d\xi\, d\tau \\ &\quad \leqslant C(\alpha) \int |\hat{u}^\varepsilon - \hat{u}|^2\, d\xi\, d\tau, \end{aligned} \tag{3.46}$$

and according to the continuity of B, this last expression converges to zero as ε tends to zero. Function $u_\alpha^\varepsilon \cdot 1_{[0,T]} = u_{\alpha,T}$ satisfies relation (3.43), which can be rewritten as

$$\begin{aligned} &(A_T^0 u_{\alpha,T}^\varepsilon, u_{\alpha,T}^\varepsilon) - \int |u_\alpha^\varepsilon(x,T)|^2\, dx \\ &\quad = (2\pi)^{-n-1} \int_{|\tau| \leqslant |\xi|} |\hat{u}_{\alpha,T}^\varepsilon(\xi,\tau)|^2 \sqrt{|\xi|^2 - \tau^2}\, d\tau\, d\xi. \end{aligned} \tag{3.47}$$

Thanks to (3.44), (3.45) and (3.46), we can pass to the limit in (3.47) as ε tends to zero, and we obtain

$$(A_T^0 u_{\alpha,T}, u_{\alpha,T}) - \int |u_\alpha(x, T)|^2 \, dx$$

$$= (2\pi)^{-n-1} \int_{|\tau| \leqslant |\xi|} |\hat{u}_{\alpha,T}(\xi, \tau)|^2 \sqrt{|\xi|^2 - \tau^2} \, d\tau \, d\xi. \tag{3.48}$$

As α decreases to zero, the left-hand side of (3.48) converges to

$$(A^0 u_T, u_T) - \int |u(x, T)|^2 \, dx;$$

to prove that the right-hand side of (3.48) converges to the right expression, one has only to notice that, for fixed ξ and τ, the sequence

$$|\hat{u}_{\alpha,T}(\xi, \tau)|^2 = |\hat{u}(\xi, \tau)|^2 (1 + \alpha |\xi|)^{-1}$$

increases; therefore, the integral expression

$$\int_{|\tau| \leqslant |\xi|} |\hat{u}_\alpha(\xi, \tau)|^2 \sqrt{|\xi|^2 - \tau^2} \, d\tau \, d\xi$$

converges to

$$\int_{|\tau| \leqslant |\xi|} |\hat{u}(\xi, \tau)|^2 \sqrt{|\xi|^2 - \tau^2} \, d\tau \, d\xi$$

which proves (3.31). ∎

To complete the proof of Theorem 6, we have only to show that A_T^0 has indeed an adjoint, and that $D(A_T^0) \cap D((A_T^0)^*)$ contains a dense subset of H. But, we can define $(A_T^0)^*$ by studying for $a > 0$

$$B_a^* v_R = \int_{\operatorname{Im} \tau = a} e^{i(t\tau + x \cdot \xi)} \hat{v}_R(x, \xi) \, \rho(|\xi|^2 - \tau^2)^{-1} \, d\xi \, d\tau,$$

where $\rho(|\xi|^2 - \tau^2)$ is the determination of $\sqrt{|\xi|^2 - \tau^2}$ taken from above, according to Fig. 3.

Then, one shows in exactly the same fashion as in the proof of Theorem 1 that B_a^* does not depend on $a > 0$, and that B^* sends continuously H to $C^0(\mathbb{R}; \mathbb{R}^n)$. Moreover, $1_{[0,T]} B^*$ is the adjoint of B_T, so that the inverse of $1_{[0,T]} B^*$ is the adjoint $(A_T)^*$. This yields the positivity of $(A_T^0)^*$, but we have in fact an equality similar to (3.31):

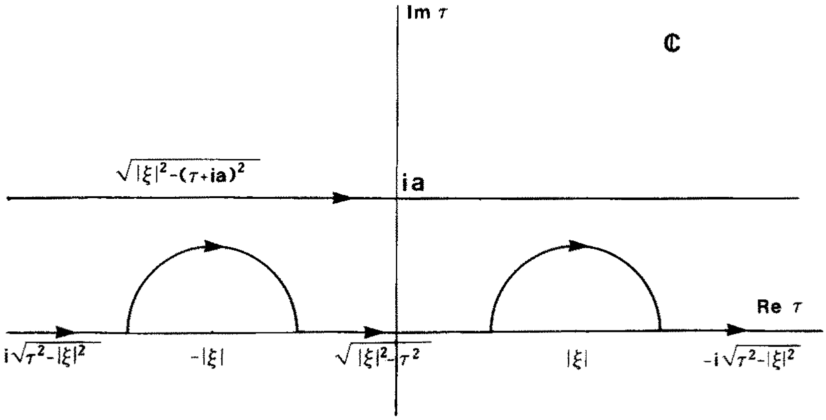

FIG. 3. The representation of the determination of $\sqrt{|\xi|^2-\tau^2}$ for the symbol of the adjoint of A.

$$((A_T^0)^*u, u) = \int_{\mathbb{R}^n} |u(x,0)|^2\,dx + (2\pi)^{-n-1}$$

$$\times \int_{|\xi|\leqslant|\tau|} |u(\xi,\tau)|^2 \sqrt{|\xi|^2-\tau^2}\,d\xi\,d\tau. \tag{3.49}$$

This ends to poof of Theorem 6.

We have the following result on A_T:

PROPOSITION 7. *A_T sends continuously*

$$V = \left\{ u \in L^2(0,T; H^{1/2}(\mathbb{R}^n)) \middle/ \frac{\partial u}{\partial t} \in L^2(0,T; H^{-1/2}(\mathbb{R}^n)) \text{ and } u(\cdot,0)=0 \right\}$$

to

$$L^2(0,T; H^{-1/2}(\mathbb{R}^n)).$$

Proof. We have only to prove that for $u_T = u \cdot 1_{[0,T]}$ and u in $\mathscr{S}(\mathbb{R}^n \times (0,\infty))$ we have the estimate

$$|A_T u_T|_{L^2(0,T;H^{-1/2}(\mathbb{R}^n))} \leqslant c\,|u_T|_V \tag{3.50}$$

for a certain finite constant c.

We define a function w as follows:

$$w(x,t) = u(x,t) \qquad \text{if} \quad 0 \leqslant t \leqslant T, x \in \mathbb{R}^n,$$

$$w(x,t) = u(x, 2T-t) \qquad \text{if} \quad T \leqslant t, \qquad x \in \mathbb{R}^n.$$

Of course, w is at most Lipschitz continuous with respect to time, but we shall see now that Aw belongs to $L^2((0, \infty); H^{-1/2}(\mathbb{R}^n))$; we have

$$w_t(x, t) = (2\pi)^{-N} \int e^{i(x \cdot \xi + t\tau)}\, i\tau\hat{w}(\xi, \tau)\, d\xi\, d\tau$$

$$\nabla \cdot w(x, t) = (2\pi)^{-N} \int e^{i(x \cdot \xi + t\tau)}\, i\xi\hat{w}(\xi, \tau)\, d\xi\, d\tau,$$

so that

$$(2\pi)^{-N} \iint |\tau\hat{w}(\xi, \tau)|^2\, d\xi\, d\tau = 2\int_0^T \int_{\mathbb{R}^n} |u_t(x, t)|^2\, dx\, dt \tag{3.51}$$

$$(2\pi)^{-N} \iint |\xi\hat{w}(\xi, \tau)|^2\, d\xi\, d\tau = 2\int_0^T \int_{\mathbb{R}^n} |\nabla u(x, t)|^2\, dx\, dt. \tag{3.52}$$

Therefore, as

$$|\sqrt{|\xi|^2 - \tau^2}\, \hat{w}(\xi, \tau)| \leqslant |\xi|\, |\hat{w}(\xi, \tau)| \qquad \text{if} \quad |\xi| \geqslant |\tau|,$$
$$|\sqrt{|\xi|^2 - \tau^2}\, \hat{w}(\xi, \tau)| \leqslant |\tau|\, |\hat{w}(\xi, \tau)| \qquad \text{if} \quad |\xi| \leqslant |\tau|,$$

we can see from (3.51) and (3.52) that Aw belongs to $L^2((0, \infty); L^2(\mathbb{R}^n))$ and in particular to $L^2((0, \infty); H^{-1/2}(\mathbb{R}^n))$. Consequently, $Aw \cdot 1_{[0,T]} \in L^2((0, T); H^{-1/2}(\mathbb{R}^n))$, and

$$Aw \cdot 1_{[0,T]} = Au \cdot 1_{[0,T]} = A_T u_T. \tag{3.53}$$

We have

$$\begin{aligned}
&|Aw|^2_{L^2((0,\infty);H^{-1/2}(\mathbb{R}^n))} \\
&\quad = |Aw|^2_{L^2(\mathbb{R};H^{-1/2}(\mathbb{R}^n))} \\
&\quad = (2\pi)^{-n-1} \int |A\hat{w}(\xi, \tau)|\, (1 + |\xi|)^{-1}\, d\xi\, d\tau \\
&\quad = (2\pi)^{-n-1} \int |\sqrt{|\xi|^2 - \tau^2}\,|^2\, |\hat{w}(\xi, \tau)|^2 (1 + |\xi|)^{-1}\, d\xi\, d\tau.
\end{aligned}$$

But we notice that

$$|\sqrt{|\xi|^2 - \tau^2}\,| \leqslant \max(|\xi|, |\tau|),$$

so that

$$|\sqrt{|\xi|^2 - \tau^2}\,|\, (1 + |\xi|)^{-1} \leqslant 1 + |\xi| + |\tau|^2 (1 + |\xi|)^{-1}$$

and as

$$(2\pi)^{-n-1}\int (1+|\xi|)\,|\hat{w}(\xi,\tau)|^2\,d\xi\,d\tau = |w|^2_{L^2(0,\infty;H^{1/2}(\mathbb{R}^n))}$$

and

$$(2\pi)^{-n-1}\int (1+|\xi|)^{-1}|\tau\hat{w}(\xi,\tau)|^2\,d\xi\,d\tau = \left|\frac{\partial w}{\partial t}\right|^2_{L^2(0,\infty;H^{-1/2}(\mathbb{R}^n))}.$$

We finally obtain

$$|Aw|^2_{L^2((0,\infty);H^{-1/2}(\mathbb{R}^n))} \leqslant |w|^2_{L^2(0,\infty;H^{1/2}(\mathbb{R}^n))} + \left|\frac{\partial w}{\partial t}\right|^2_{L^2(0,\infty;H^{-1/2}(\mathbb{R}^n))},$$

and according to (3.51)–(3.53),

$$|A_T u_T|_{L^2((0,T);H^{-1/2}(\mathbb{R}^n))} \leqslant 2\,|u|_V,$$

which proves (3.50). By density, we obtain Proposition 7. ∎

Proposition 7 allows us to identify $A_T^{1}|_V$ and A_T^{1} defined in (2.13); we can see that $D(A_T^{1}) \equiv V$.

We have now a lemma relative to the positivity of A_T in V:

LEMMA 8. *For u in V we have the identity*

$$(A_T u, u) = (2\pi)^{-n-1}\int_{|\tau|\leqslant|\xi|} |\hat{u}(\xi,\tau)|^2\sqrt{|\xi|^2-\tau^2}\,d\xi\,d\tau + \int |u(x,T)|^2\,dx.$$

Proof. Let u be given in V, and let $v = A_T u$; define, for $\varepsilon > 0$, a function v^ε by

$$\hat{v}^\varepsilon = (1+\varepsilon|\xi|^2)^{-1/2}\hat{v}.$$

Then v^ε belongs to H for every $\varepsilon > 0$, and if we set $u^\varepsilon = B_T v^\varepsilon$, the following identity is true:

$$(A_T u^\varepsilon, u^\varepsilon) = (2\pi)^{-n-1}\int_{|\tau|\leqslant|\xi|} |\hat{u}^\varepsilon(\xi,\tau)|^2\sqrt{|\xi|^2-\tau^2}\,d\xi\,d\tau + \int |u^\varepsilon(x,T)|^2\,dx. \tag{3.54}$$

But

$$\int_{|\tau|\leqslant|\xi|} |u^\varepsilon(\xi,\tau)|^2 \sqrt{|\xi|^2-\tau^2}\, d\xi\, d\tau$$

$$= \int_{|\tau|\leqslant|\xi|} |\hat{u}(\xi,\tau)|^2 (1+\varepsilon|\xi|^2)^{-1} \sqrt{|\xi|^2-\tau^2}\, d\xi\, d\tau,$$

and similarly,

$$\int |u^\varepsilon(x,T)|^2\, dx = (2\pi)^{-n} \int |(\mathscr{F}_x u^\varepsilon)(\xi,T)|^2\, d\xi$$

$$= (2\pi)^{-n} \int |(\mathscr{F}_x u)(\xi,T)|^2 (1+\varepsilon|\xi|^2)^{-1}\, d\xi.$$

Notice that every element of V can be identified with a function from $[0, T]$ to $L^2(\mathbb{R}^n)$ which is weakly continuous in t; thus $u(\cdot, T)$ is uniquely defined.

The left-hand side of (3.54) converges to $\mathrm{Re}(A_T u, u)$ and the integrands on the right-hand side are increasing as ε tends to zero. Therefore, the right-hand side of (3.54) converges to

$$(2\pi)^{-n-1} \int_{|\tau|\leqslant|\xi|} |\hat{u}(\xi,\tau)|^2 \sqrt{|\xi|^2-\tau^2}\, d\xi\, d\tau + \int |(x,T)|^2\, dx,$$

which proves Lemma 8. ∎

4. Existence, Uniqueness and Energy Properties for the Unilateral Problem

The results of Section 3 enable us to show that problem (2.18) possesses a solution. Together with Theorem 0, and a regularity result, this will prove that (2.1)–(2.4) possesses an energy conserving solution.

THEOREM 9. *For any φ in $L^2((0,T); H^{1/2}(\mathbb{R}^n))$ such that $\partial\varphi/\partial t$ belongs to $L^2((0,T); H^{-1/2}(\mathbb{R}^n))$, there exists a unique function v in*

$$V = \left\{ u \in L^2(0,T; H^{1/2}(\mathbb{R}^n)) \,\middle/\, \frac{\partial u}{\partial t} \in L^2(0,T; H^{-1/2}(\mathbb{R}^n)) \right\},$$

which satisfies the following relations:

$$A_T^1 v \geqslant \varphi \tag{4.1}$$

$$v \geqslant 0 \tag{4.2}$$

$$\langle A_T^1 v - \varphi, v\rangle = 0. \tag{4.3}$$

Proof. Let $r^- = -\min(r, 0)$, and approximate system (4.1)–(4.3) by a penalty method:

$$A_T^1 v^\varepsilon - (1/\varepsilon)(v^\varepsilon)^- = \varphi. \tag{4.4}$$

Equation (4.4) indeed has a solution; if we write it as

$$\square u^\varepsilon = 0 \qquad \text{in } Q_T, \tag{4.5}$$

$$u^\varepsilon(X, 0) = u_t^\varepsilon(X, 0) = 0 \qquad \text{in } \Omega, \tag{4.6}$$

$$-\frac{\partial u_\varepsilon}{\partial y} - \frac{1}{\varepsilon}(v^\varepsilon)^- = \varphi \qquad \text{on } \Sigma_T, \tag{4.7}$$

it appears as the perturbation of

$$\square W = 0 \qquad \text{in } Q_T,$$

$$W(X, 0) = W_t(X, 0) = 0 \qquad \text{in } \Omega,$$

$$-\frac{\partial W}{\partial y} = \varphi \qquad \text{on } \Sigma_T,$$

by the Lipschitz continuous nonlinearity $r \to -\varepsilon^{-1} r^-$. Therefore, (4.5) – (4.7) can be proved to have a solution by standard techniques, such as successive iterations.

If we multiply (4.5) by $\partial u^\varepsilon / \partial t$ and integrate on Q_T for $t \leqslant T$, we have

$$\frac{1}{2}\int_\Omega \left(\left| \frac{\partial u_\varepsilon}{\partial t}(X, t) \right|^2 + |\nabla u^\varepsilon(X, t)|^2 \right) dX - \int_{\Sigma_t} \frac{\partial u_\varepsilon}{\partial y} \frac{\partial u_\varepsilon}{\partial s}\, dx\, ds = 0,$$

and, substituting (4.7) and integrating by parts, we obtain

$$\frac{1}{2}\int_\Omega \left(\left| \frac{\partial u_\varepsilon}{\partial t}(X, t) \right|^2 + |\nabla u^\varepsilon(X, t)|^2 \right) dX + \frac{1}{2\varepsilon}\int_\Gamma [(u^\varepsilon)^-]^2(x, t)\, dx$$

$$= \int_\Gamma \varphi(x, t)\, u^\varepsilon(x, t)\, dx - \int_{\Sigma_t} \frac{\partial \varphi}{\partial t} u^\varepsilon\, dx\, ds. \tag{4.8}$$

According to the functional hypothesis on φ, we get through a Gronwall's lemma the following estimates:

$$\frac{\partial u^\varepsilon}{\partial t} \text{ and } u^\varepsilon \text{ are bounded in } L^\infty((0, T); L^2(\Omega)) \text{ uniformly with respect to } \varepsilon; \tag{4.9}$$

$$\frac{1}{\varepsilon}\int_\Gamma |(u^\varepsilon)^-|^2(x, t)\, dx \leqslant c \text{ uniformly with respect to } \varepsilon. \tag{4.10}$$

We can deduce from the above statement (4.9) that if $v^\varepsilon = u^\varepsilon|_\Sigma$, then

$$v^\varepsilon \text{ is bounded in } L^\infty((0,T); H^{1/2}(\mathbb{R}^n)) \text{ uniformly with respect to } \varepsilon; \tag{4.11}$$

$$\frac{\partial v_\varepsilon}{\partial t} \text{ is bounded in } L^\infty((0,T); H^{-1/2}(\mathbb{R}^n)) \text{ uniformly with respect to } \varepsilon. \tag{4.12}$$

From (4.11), (4.12) and Proposition 7, we deduce that

$$A_T^{\frac{1}{2}} v^\varepsilon \text{ is bounded in } L^2([0,T]; H^{-1/2}(\mathbb{R}^n)) \text{ uniformly with respect to } \varepsilon. \tag{4.13}$$

We may extract from the sequence $(v^\varepsilon)_\varepsilon$ a subsequence, still denoted by $(v^\varepsilon)_\varepsilon$, and such that

$$\begin{aligned} v^\varepsilon &\rightharpoonup v && \text{in } L^\infty((0,T; H^{1/2}(\mathbb{R}^n)) \text{ weak}*, \\ \frac{\partial v_\varepsilon}{\partial t} &\rightharpoonup \frac{\partial v}{\partial t} && \text{in } L^\infty((0,T); H^{-1/2}(\mathbb{R}^n)) \text{ weak}*, \\ A_T^{\frac{1}{2}} v^\varepsilon &\rightharpoonup A_T v && \text{in } L^2((0,T); H^{-1/2}(\mathbb{R}^n)) \text{ weak}. \end{aligned} \tag{4.14}$$

Let

$$\mu^\varepsilon = (v^\varepsilon)^- \varepsilon^{-1}, \tag{4.15}$$

then

$$\mu^\varepsilon \to A_T^{\frac{1}{2}} v - \varphi \qquad \text{in } L^2((0,T); H^{-1/2}(\mathbb{R}^n)) \text{ weak}.$$

We set

$$\mu = A_T^{\frac{1}{2}} v - \varphi. \tag{4.16}$$

From (4.15),

$$\mu \geqslant 0 \qquad (\text{i.e., } \langle \mu, \varphi \rangle \geqslant 0 \text{ for all nonnegative test function } \varphi), \tag{4.17}$$

and from (4.10)

$$v \geqslant 0. \tag{4.18}$$

We shall now show, with the help of (now) classical monotonicity techniques, that

$$(\mu, v) = 0. \tag{4.19}$$

Notice first that (μ, v) is defined because μ is the sum of $-\varphi$ which belongs to $L^2((0, T); H^{1/2}(\mathbb{R}^n))$ and $A_T^1 v$ which belongs to $L^2((0, T); H^{-1/2}(\mathbb{R}^n))$, so that μ is in the dual of $L^2((0, T); H^{1/2}(\mathbb{R}^n))$.

As A_T^1 is positive (see Lemma 8),

$$(A_T^1 v^\varepsilon - A_T^1 v, v^\varepsilon - v) \geqslant 0,$$

so that

$$(A_T^1 v^\varepsilon, v^\varepsilon) \geqslant (A_T^1 v, v^\varepsilon - v) + (A_T^0 v, v),$$

and in the limit

$$\underline{\lim}\, (A_T^1 v^\varepsilon, v^\varepsilon) \geqslant (A_T^1 v, v). \tag{4.20}$$

On the other hand, we have

$$(A_T^1 v^\varepsilon, v - v^\varepsilon) - (1/\varepsilon)((v^\varepsilon)^-, v - v^\varepsilon) = (\varphi, v - v^\varepsilon),$$

and in virtue of (4.18),

$$(A_T^1 v^\varepsilon, v - v^\varepsilon) - (1/\varepsilon)((v^\varepsilon)^- - v^-, v - v^\varepsilon) = (\varphi, v - v^\varepsilon).$$

Now, we know that $(s^- - r^-)(r - s) \geqslant 0$ for all real numbers r and s so that

$$(A_T^1 v^\varepsilon, v^\varepsilon) \leqslant (A_T^1 v^\varepsilon, v) + (\varphi, v^\varepsilon - v).$$

In the limit we have thus

$$\overline{\lim}\, (A_T^1 v^\varepsilon, v^\varepsilon) \leqslant (A_T^1 v, v)$$

which gives with (4.20)

$$\lim_{\varepsilon \to 0} (A_T^1 v^\varepsilon, v^\varepsilon) = (A_T^1 v, v). \tag{4.21}$$

Let w be an arbitrary nonnegative function chosen in $L^2((0, T); H^{1/2}(\mathbb{R}^n))$. Then

$$(A_T^1 v^\varepsilon, w - v^\varepsilon) - (1/\varepsilon)((v^\varepsilon)^-, w - v^\varepsilon) = (\varphi, w - v^\varepsilon),$$

so that, as w is nonnegative,

$$(A_T^1 v^\varepsilon, w - v^\varepsilon) \geqslant (\varphi, w - v^\varepsilon). \tag{4.22}$$

We can now pass to the limit easily in (4.22), and we obtain, with the help of (4.21),

$$(A_T^1 v, w - v) \geqslant (\varphi, w - v), \qquad \forall w. \tag{4.23}$$

From the definition (4.16) of μ,

$$(\mu, w - v) \geqslant 0, \qquad \forall\, w \in L^2((0, T); H^{1/2}(\mathbb{R}^n)) \text{ such that } w \geqslant 0.$$

If we choose $w = 0$ and $w = 2v$, we obtain

$$(\mu, w) = 0,$$

which is (4.19) and completes the existence proof.

To prove uniqueness, let t be given in $(0, T]$, and let v, v' be two solutions of (4.1)–(4.3). Then, they satisfy (4.23), i.e., $\forall\, w \in L^2((0, T); H^{1/2}(\mathbb{R}^n))$ such that $w \geqslant 0$,

$$(A_t^1 v, w - v) \geqslant (\varphi, w - v), \tag{4.24}$$

$$(A_t^1 v', w - v') \geqslant (\varphi, w - v'), \tag{4.25}$$

where here v, w and v' mean the restrictions of these functions to the interval $[0, t]$ by a slight abuse of notations.

Take in (4.24) $w = v'$ and in (4.25) $w = v$, and add the resulting inequalities to obtain

$$(A_t^1(v - v'), (v - v')) \leqslant 0.$$

Lemma 8 implies that

$$\int |(v - v')(x, t)|^2 \, dx \leqslant 0,$$

which yields the uniqueness because t is arbitrary in $[0, T]$. ∎

Remark 10. The result of Theorem 9 does no appear to be easily and directly deducible from theorems on variational inequalities. One could try, for instance, to approximate (4.1)–(4.3) by the variational inequality

$$v^m \in K^m$$

$$(A_T^1 v^m - \varphi, w - v^m) \geqslant 0, \qquad \forall\, w \in K^m,$$

where K^m is a convex set of positive functions in V, such that K^m is bounded for the norm of V. Theorem 8.1 of Chapter 2 of [4] shows the existence of such a v^m. But to obtain the existence of a limit of the sequence v^m, some amount of work is needed, because A is not V-coercive; this amount of work is about the same as that used for proving Theorem 9.

An alternative would be to use elliptic regularization such as in Theorem 6.1 of Chapter 3 of Ref. [4]. Then the operator A_T^1 has the required properties, thanks to our Theorem 6; but we miss the coercivity of A_T^1, or of the nonlinearity, and we would have the same difficulty as above. In any case, the theorem just quoted concerns inequalities of the form

$$(A_T^1 v - \varphi, w - v) \geqslant 0, \qquad \forall v \in K \cap D(A_T^1),$$

which is weaker than the form (4.1)–(4.3).

Another try would be the results on the sum of monotone operators of [3]; the required condition is the following one:

$$|(Eu, v)|^2 \leqslant C(Eu, u)(|v|_H^2 + |Ev|_H^2), \qquad \forall u, v \in D(E). \tag{*}$$

This means that E does not rotate too much u in Hilbert space. Let us check that if $E = A_T^0$, this requirement cannot be satisfied. Let first $n = 0$, so that A_T^0 can be explicitly given, according to Section 1:

$$(A_T^0 \varphi)(t) = \varphi'(t).$$

Thus, if we choose φ and ψ in $\mathscr{D}(\mathbb{R})$ with the behavior pictured in Fig. 4 below, we can see that, as $\varphi(T) = 0$, then

$$(A_T^0 \varphi, \varphi) = \int_0^T (\varphi\varphi')(t)\, dt = \tfrac{1}{2}(\varphi(T))^2 = 0,$$

and thus (*) cannot hold, as its left-hand side does not vanish.

In the case $n \geqslant 1$, choose

$$v_\varepsilon(x, t) = \varphi(t)\, a_\varepsilon(x)$$
$$w_\varepsilon(x, t) = \psi(t)\, a_\varepsilon(x),$$

with

$$a_\varepsilon(x) = \varepsilon^{n/2} \exp(-|x|^2\, \varepsilon^2).$$

Then

$$\hat{a}_\varepsilon(\xi) = \pi^{n/2} \varepsilon^{-n/2} \exp(-|\xi|^2/4\varepsilon^2);$$

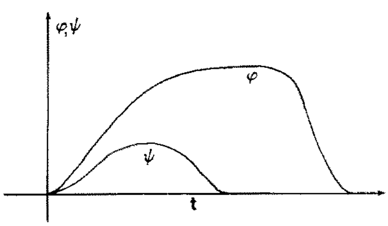

FIG. 4. The behavior of ϕ and ψ.

we can evaluate $(A_T^0 v, w)$ and $(A_T^0 v, v)$:

$$(A_T^0 v_\varepsilon, w_\varepsilon) = (2\pi)^{-n} \int \sqrt{|\xi|^2 - \tau^2}\, \hat{v}(\xi, \tau)\, \bar{\hat{w}}(\xi, \tau)\, d\xi\, d\tau$$

$$= (2\pi)^{-n} \int \hat{\varphi}(\tau)\, \bar{\hat{\psi}}(\tau) \left(\int \sqrt{|\xi|^2 - \tau^2}\, |\hat{a}_\varepsilon(\xi)|^2\, d\xi \right) d\tau.$$

For all ξ and τ, we have the inequality

$$|\sqrt{|\xi|^2 - \tau^2} - i\tau| \leqslant |\xi|$$

and we can thus evaluate

$$\left| \int \sqrt{|\xi|^2 - \tau^2}\, |\hat{a}_\varepsilon(\xi)|^2\, d\xi - \int i\tau\, |\hat{a}_\varepsilon(\xi)|^2\, d\xi \right|$$

$$\leqslant \int |\xi|\, |\hat{a}_\varepsilon(\xi)|^2\, d\xi = c_{n-1} \int_0^\infty r^n \exp(-r^2/2\varepsilon^2)\, dr\, \varepsilon^{-n} \pi^n,$$

where c_{n-1} is the $(n-1)$-dimensional measure of the sphere S^{n-1}. We know that

$$c_n = 2\pi^{(n+1)/2} \left| \Gamma\left(\frac{n+1}{2}\right) \right|^{-1}.$$

On the other hand,

$$\int_0^\infty r^n \exp\, (-r^2/2\varepsilon^2)\, dr = \varepsilon^{n+1} 2^{(n-1)/2} \int_0^\infty s^{(n-1)/2}\, e^{-s}\, ds$$

$$= \varepsilon^{n+1} 2^{(n-1)/2} \Gamma\left(\frac{n+1}{2}\right).$$

Therefore,

$$\int |\xi|\, |\hat{a}_\varepsilon(\xi)|^2\, d\xi = 2^{(n+1)/2} \pi^{(3n+1)/2} \varepsilon.$$

In the same fashion, we can compute

$$\int |\hat{a}_\varepsilon(\xi)|^2\, d\xi = \pi^n \varepsilon^{-n} \pi^{n/2} (\varepsilon \sqrt{2})^n = \pi^{3n/2} 2^{n/2}.$$

Finally,

$$\left| \int \sqrt{|\xi|^2 - \tau^2}\, |\hat{a}_\varepsilon(\xi)|^2\, d\xi + i\tau \pi^{3n/2} 2^{n/2} \right| \leqslant 2^{(n+1)/2} \pi^{(3n+1)/2} \varepsilon,$$

so that

$$\lim_{\varepsilon\to 0} (A_T^0 v_\varepsilon, w_\varepsilon) = (\pi/2)^{n/2} \int -i\tau\hat{\varphi}(\tau)\,\bar{\hat{\psi}}(\tau)\,d\tau$$

$$= (\pi/2)^{n/2} \int \varphi'(t)\,\psi(t)\,dt,$$

and

$$\lim_{\varepsilon\to 0} (A_T^0 v_\varepsilon, v_\varepsilon) = 0.$$

Let us evaluate now $|w_\varepsilon|_H^2 + |A_T^0 w_\varepsilon|_H^2$:

$$|w_\varepsilon|_H^2 = \int |a_\varepsilon(x)|^2\,dx \int_0^T |\psi(t)|^2\,dt = \left(\frac{\pi}{2}\right)^{n/2} \int |\psi(t)|^2\,dt$$

$$|A_T^0 w_\varepsilon|_H^2 \leqslant \int_{\mathbb{R}^n\times\mathbb{R}} |A w_\varepsilon|^2\,dx\,dt$$

$$= \int |\sqrt{|\xi|^2-\tau^2}|^2\,|a_\varepsilon(\xi)|^2\,|\psi(\tau)|^2\,d\xi\,d\tau$$

$$\leqslant \int (|\xi|^2+\tau^2)\,|a_\varepsilon(\xi)|^2\,|\psi(\tau)|^2\,d\xi\,d\tau$$

$$\leqslant \int |\psi(\tau)|^2\,d\tau \left(\tau^2 \int |a_\varepsilon(\xi)|^2\,d\xi \right.$$

$$\left. + \int |\xi|^2\,|a_\varepsilon(\xi)|^2\,d\xi\right)$$

$$= \int |\psi(\tau)|^2 \left(\tau^2 \left(\frac{\pi}{2}\right)^{n/2}\right.$$

$$\left. + \left[\varepsilon^2 \pi^{(3n+1)/2} 2^{(n+2)/2} \Gamma\left(\frac{n+2}{2}\right)\Big/\Gamma\left(\frac{n+1}{2}\right)\right]\right) d\tau.$$

Therefore, the right-hand side of $(*)$ tends to zero and the left-hand side of $(*)$ tends to a positive number as ε tends to zero. This proves that $(*)$ cannot be satisfied if $E = A_T^0$. ∎

We have a regularity result:

Proposition 11. *Assume that φ belongs to $H^1(\mathbb{R}^n \times (0, T))$. Then, the solution of* (4.1)–(4.3) *belongs to* $H^1(\mathbb{R}^n \times (0, T))$.

Proof. Let

$$v_h(x, t) = v(x + he_i, t),$$

where e_i is the ith vector of the basis of $\mathbb{R}^n$. Then we have, from (4.23),

$$(A_t^1 v - \varphi, w - v) \geqslant 0$$

$$(A_t^1 v_h - \varphi_h, w - v_h) \geqslant 0.$$

If we take $w = v_h$ in the first of these two inequalities, then

$$(A_t^1(v_h - v), v_h - v) \leqslant (\varphi_h - \varphi, v_h - v);$$

we divide the above inequality by h, and with the help of Lemma 8, we obtain

$$\int_\Gamma \left| \frac{(v_h - v)(x, t)}{h} \right|^2 dx \leqslant \int_{\Sigma_t} \left| \frac{v_h - v}{h} \frac{\varphi_h - \varphi}{h} \right| (x, s)\, dx\, ds; \qquad (4.26)$$

let

$$\int_\Gamma \int_0^t \left| \frac{(v_h - v)(x, s)}{h} \right|^2 dx\, ds = F(t);$$

then, (4.26) gives

$$F'(t) \leqslant \left(\int_\Gamma \int_0^t \left| \frac{\varphi_h - \varphi}{h} \right|^2 dx\, ds \right)^{1/2} \sqrt{F(t)}$$

$$\leqslant |\varphi|_{H^1(\mathbb{R}^n \times (0, T))} \sqrt{F(t)},$$

and it follows that

$(v_h - v)/h$ is bounded in $L^\infty((0, T); L^2(\mathbb{R}^n))$ independently from h.

Passing to the limit as h tends to zero, we get

$$\frac{\partial v}{\partial x_i} \in L^\infty((0, T); L^2(\mathbb{R}^n)), \qquad \forall i \in \{1, \ldots, n\}.$$

For the time derivative, we proceed in an analogous fashion, but more carefully to take the boundaries into account. We thus obtain Proposition 11, and even the information

$$v \in L^\infty((0, T); H^1(\mathbb{R}^n)) \qquad (4.27)$$

$$\frac{\partial v}{\partial t} \in L^\infty((0, T); L^2(\mathbb{R}^n)). \quad \blacksquare \qquad (4.28)$$

With the help of this regularity result, we can prove energy conservation.

THEOREM 12. *Let φ be given in $H^1(\mathbb{R}^n \times (0, T))$, and let v be the solution of* (4.1)–(4.3). *Let z satisfy*

$$\Box z = 0 \qquad \text{in } Q_T, \tag{4.29}$$

$$z(X, 0) = z_t(X, 0) = 0 \qquad \text{in } \Omega, \tag{4.30}$$

$$z(, 0, t) = v(x, t) \qquad \text{on } \Sigma_T. \tag{4.31}$$

Then z satisfies, for all t between 0 *and T, the following energy equality*:

$$\frac{1}{2}\int \left\{ \left| \frac{\partial z}{\partial t}(X, t) \right|^2 + |\nabla z(X, t)|^2 \right\} dX$$

$$= -\int_{\Sigma_t} \frac{\partial \varphi}{\partial t}(x, s)\, v(x, s)\, dx\, ds + \int_\Gamma \varphi(x, t)\, v(x, t)\, dx. \tag{4.32}$$

Proof. With the smoothness assumptions that we made on φ, we have

$$v \in L^\infty(0, T; H^1(\mathbb{R}^n))$$

$$\frac{\partial v}{\partial t} \in L^\infty(0, T; L^2(\mathbb{R}^n))$$

$$\frac{\partial v}{\partial n} \in L^2(0, T; L^2(\mathbb{R}^n)).$$

Therefore, z is of bounded energy for each time t. If we multiply (4.28) by z_t and integrate on Q_t, we obtain

$$\frac{1}{2}\int \left(\left| \frac{\partial z}{\partial t}(X, t) \right|^2 + |\nabla z(X, t)|^2 \right) dX$$

$$= -\int_{\Sigma_t} \frac{\partial z}{\partial y}(x, 0, s) \frac{\partial z}{\partial t}(x, 0, s)\, dx\, ds.$$

We then notice that

$$-\frac{\partial z}{\partial y} = A_t^1 v = \varphi + \mu,$$

and z belongs to $L^2(\mathbb{R}^n \times (0, T))$; $\mu \geqslant 0$, $v \geqslant 0$ and

$$\int_{\Sigma_t} \mu(x, s)\, v(x, s)\, dx\, ds = 0.$$

In particular,

$$\mu = 0 \qquad \text{a.e. on } \{(x, t)/v(x, t) > 0\},$$

so that

$$\int_{\Sigma_t} \frac{\partial v}{\partial t} \mu \, dx \, ds = \int_{\Sigma_t, v=0} \frac{\partial v}{\partial t} \mu \, dx \, ds.$$

But

$$\frac{\partial v}{\partial t} = 0 \qquad \text{a.e. on } \{(x, t)/v(x, t) = 0\},$$

whence it follows that

$$\int_{\Sigma_t} \mu(x, s) \frac{\partial v}{\partial t} (x, s) \, dx \, ds = 0.$$

We conclude now that

$$\int_{\Sigma_t} A_t^{\frac{1}{2}} v \frac{\partial v}{\partial t} \, dx \, ds = \int_{\Sigma_t} \varphi \frac{\partial v}{\partial t} \, dx \, ds,$$

which yields (4.31) by integration by parts.

We have a corollary to Theorem 12:

COROLLARY 13. *Let u_0 be given in $H_0^1(\Omega) \cap H^{3/2}(\Omega)$, u_1 in $H^{1/2}(\Omega)$ and f in $H^{3/2}(Q_T)$. Then the problem*

$$\Box u = f \qquad \text{in } Q_T, \tag{4.33}$$

$$u(X, 0) = u_0(X) \qquad \text{in } \Omega, \tag{4.34}$$

$$u_t(X, 0) = u_1(X) \qquad \text{in } \Omega, \tag{4.35}$$

$$\left.\begin{aligned} u(x, 0, t) &\geqslant 0 \\ \frac{\partial u}{\partial n} (x, 0, t) &\geqslant 0 \\ u(x, 0, t) \cdot \frac{\partial u}{\partial n} (x, 0, t) &= 0 \end{aligned}\right\} \qquad \text{on } \Sigma_T, \tag{4.36}$$

possesses a unique solution u in $L^\infty(0, T; H^{3/2}(\Omega)) \cap W^{1,\infty}(0, T; H^{1/2}(\Omega))$ which satisfies the identity, for all t,

$$\frac{1}{2} \int \left(\left| \frac{\partial u}{\partial t} (X, t) \right|^2 + |\nabla u(X, t)|^2 \right) dX - \frac{1}{2} \int (|u_1|^2 + |\nabla u_0|^2) \, dX$$

$$= \int_{Q_t} f \frac{\partial u}{\partial t} \, dX \, ds. \tag{4.37}$$

Proof. From Theorem 0, Proposition 11 and Theorem 12, we know the existence of u in the space $L^\infty(0, T; H^{3/2}(\Omega)) \cap W^{1,\infty}(0, T; H^{1/2}(\Omega))$. We have only to check (4.36). If we multiply (4.32) by $\partial u/\partial t$ and integrate on Q_t, we obtain

$$\frac{1}{2}\int\left(\left|\frac{\partial u}{\partial t}(X,t)\right|^2 + |\nabla u(X,t)|^2\right) dX - \frac{1}{2}\int(|u_1|^2 + |\nabla u_0|^2)\, dX$$
$$= \int_{Q_t} f u_t \, dX\, ds + \int_{\Sigma_t}\left(\frac{\partial u}{\partial n}\frac{\partial u}{\partial t}\right)(x, 0, s)\, dx\, ds.$$

We can check immediately that, thanks to (4.35), the term

$$\int_{\Sigma_t}\left(\frac{\partial u}{\partial n}\frac{\partial u}{\partial t}\right)(x, 0, s)\, dx\, ds$$

vanishes, and this gives exactly (4.36). ∎

We would like to get rid of the restriction $u_0|_\Gamma = 0$; if we assume some smoothness and let u_0 satisfy

$$u_0|_\Gamma \geqslant 0$$
$$\left.\frac{\partial u_0}{\partial n}\right|_\Gamma \geqslant 0$$
$$u_0 \cdot \left.\frac{\partial u_0}{\partial n}\right|_\Gamma = 0,$$

then this is possible by the same reduction argument as in Section 2, if w is the solution of

$$\begin{aligned}
\Box w &= f && \text{in } Q_T,\\
w(X, 0) &= u_0(X) && \text{in } \Omega,\\
w_t(X, 0) &= u_1(X) && \text{in } \Omega,\\
w(x, 0, t) &= u(x, 0) && \text{on } \Sigma.
\end{aligned}$$

Instead of (2.18), we have then to solve

$$(Av)(x,t) + \beta(v(x,t) + u_0(x, 0)) \ni \frac{\partial w}{\partial y}(x, 0, t).$$

This is achieved by the same means as before, and yields a result similar to Theorem 9, but we can have the energy relation (4.36) only if we know that $(\partial u/\partial n)(x, 0, t)$ and $(\partial u/\partial t)(x, 0, t)$ are square integrable. This will be the case only if $(\partial w/\partial y)(x, 0, t) + (A_T^1 u_0)(x, t)$ belongs to $H^1(\mathbb{R}^n \times (0, T))$, in which case we have the same conclusion as in Proposition 11. We leave the exact smoothness requirement to the reader.

5. Can We Generalize to a Domain of Arbitrary Shape?

We shall first start this section with a result concerning slab-shaped domains. We have the following theorem:

Theorem 14. *Let $\Omega = \mathbb{R}^n \times (0, L)$ and let there be given u_0 in $H_0^1(\Omega) \cap H^{3/2}(\Omega)$, u_1 in $H^{1/2}(\Omega)$, f in $H^{3/2}(Q_T)$. Then the problem*

$$\Box u = f \qquad \text{in } Q_T, \tag{5.1}$$

$$u(X, 0) = u_0(X) \qquad \text{in } \Omega, \tag{5.2}$$

$$u_t(X, 0) = u_1(X) \qquad \text{in } \Omega, \tag{5.3}$$

$$\left.\begin{aligned} &u \geqslant 0 \\ &\frac{\partial u}{\partial n} \geqslant 0 \\ &u \cdot \frac{\partial u}{\partial n} = 0 \end{aligned}\right\} \qquad \text{on } \Sigma_T = (\{0\} \cup \{L\}) \times \mathbb{R}^n \times (0, T) \tag{5.4}$$

possesses a unique solution which satisfies the energy identity, for all t,

$$\begin{aligned} &\frac{1}{2}\int \left(\left| \frac{\partial u}{\partial t}(X, t) \right|^2 + |\nabla u(X, t)|^2 \right) dX - \frac{1}{2}\int (|u_1|^2 + |\nabla u_0|^2)\, dX \\ &= \int_{Q_t} f \frac{\partial u}{\partial t}\, dX\, ds. \end{aligned} \tag{5.5}$$

Proof. Let w be the solution of

$$\begin{aligned} \Box w &= f && \text{in } Q_T, \\ w(X, 0) &= u_0(X) && \text{in } \Omega, \\ w_t(X, 0) &= u_1(X) && \text{in } \Omega, \\ w|_{\Sigma_T} &= 0. \end{aligned}$$

Then, we have to solve, if $u = z + w$,

$$\begin{aligned} \square z &= 0 && \text{in } Q_T, \\ z(X, 0) &= 0 && \text{in } \Omega, \\ z_t(X, 0) &= 0 && \text{in } \Omega, \end{aligned}$$

$$\left.\begin{aligned} & z \geqslant 0 \\ & \frac{\partial z}{\partial n} \geqslant -\frac{\partial w}{\partial n} \\ & z \cdot \left(\frac{\partial z}{\partial n} + \frac{\partial w}{\partial n}\right) = 0 \end{aligned}\right\} \quad \text{on } \Sigma_T.$$

Thanks to the propagation properties of the wave equation, the function z vanishes identically in the set

$$\{(X, t)/0 \leqslant t \leqslant L/2 - |y - L/2|\}.$$

Therefore, we shall solve for $T = L/2$ the problem

$$A_T^1 \bar{v} + \beta(\bar{v}) \ni \frac{\partial w}{\partial n}(x, 0, t),$$

and similarly,

$$A_T^1 \tilde{v} + \beta(\tilde{v}) \ni \frac{\partial w}{\partial n}(x, L, t).$$

Clearly $\bar{v}$ and $\tilde{v}$ exist, and if $\bar{z}$ is defined by

$$\begin{aligned} \square \bar{z} &= 0 && \text{in } \mathbb{R}^n \times (0, \infty) \times (0, L/2), \\ \bar{z}(X, 0) &= \bar{z}_t(X, 0) = 0 && \\ \bar{z}(x, 0, t) &= \bar{v}(x, t) && \text{on } \mathbb{R}^n \times (0, L/2), \end{aligned}$$

and if analogously $\tilde{z}$ is defined by

$$\begin{aligned} \square \tilde{z} &= 0 && \text{in } \mathbb{R}^n \times (-\infty, L) \times (0, L/2), \\ \tilde{z}(X, 0) &= \tilde{z}_t(X, 0) = 0 && \\ \tilde{z}(x, L, t) &= \tilde{v}(x, t) && \text{on } \mathbb{R}^n \times (0, L/2), \end{aligned}$$

then we can observe that $\bar{z}$ vanishes on the set $\{(X, t)/0 \leqslant t \leqslant y\}$ and that $\tilde{z}$ vanishes on the set $\{(X, t)/0 \leqslant t \leqslant L - y\}$; therefore, it is a routine matter to check that $z = \bar{z} + \tilde{z}$ solves (5.1) – (5.4) on $\mathbb{R}^n \times (0, L) \times (0, L/2)$. Then it is easy to show that (5.5) is satisfied. ∎

In the same fashion, one may replace condition (5.4) by

$$\begin{aligned} u &\geqslant 0 \\ \frac{\partial u}{\partial n} &\geqslant 0 \qquad \text{on } \{0\} \times \mathbb{R}^n \times (0, T), \\ u \cdot \frac{\partial u}{\partial n} &= 0 \end{aligned} \tag{5.4'}$$

and

$$u(x, L, t) = 0 \qquad \text{on } \mathbb{R}^n \times (0, T). \tag{5.4''}$$

The conclusion is then the same as in Theorem 14. One could very well replace (5.4″) by a Neumann condition, instead of a Dirichlet condition; then, one has only to assume that $u_0 = 0$ at $x = 0$, and the conclusions of Theorem 14 still hold.

Nevertheless, we have the following important fact:

THEOREM 15. *Let z belong to $H^1(Q_T)$ and satisfy*

$$\begin{aligned} &\Box z = 0 && \text{in } Q_T, \\ &z(X, 0) = z_t(X, 0) = 0 && \text{in } \Omega, \\ &z(x, 0, t) = \varphi(x, t) && \text{on } \Sigma_T, \\ &z(x, L, t) = \psi(x, t) && \text{on } \Sigma_T. \end{aligned} \tag{5.6}$$

Then

$$\int_{\Sigma_t} z \frac{\partial z}{\partial n}\, dx\, dt \geqslant 0, \qquad \forall \varphi, \psi,$$

if and only if $T \leqslant L$.

Proof. (if): Let $\bar{z}$ be the solution of

$$\begin{aligned} &\Box \bar{z} = 0 && \text{in } \mathbb{R}^n \times (0, \infty) \times (0, T), \\ &\bar{z}(X, 0) = \bar{z}_t(X, 0) = 0 && \text{in } \mathbb{R}^n \times (0, \infty), \\ &\bar{z}(x, 0, t) = \varphi(x, t) && \text{on } \mathbb{R}^n \times (0, T), \end{aligned}$$

and let $\tilde{z}$ be the solution of

$$\begin{aligned} &\Box \tilde{z} = 0 && \text{in } \mathbb{R}^n \times (-\infty, L) \times (0, T), \\ &\tilde{z}(X, 0) = \tilde{z}_t(X, 0) = 0 && \text{in } \mathbb{R}^n \times (-\infty, L), \\ &\tilde{z}(x, L, t) = \psi(x, t) && \text{on } \mathbb{R}^n \times (0, T). \end{aligned}$$

Then, if $T \leqslant L$, $z = \tilde{z} + \bar{z}$ is the solution of (5.6), thanks to the propagation properties of the wave equation. But

$$\int_{\Sigma_T} z \frac{\partial z}{\partial n} \, dx \, dt = \int_{\Sigma_T} (\bar{z} + \tilde{z}) \left(\frac{\partial \bar{z}}{\partial n} + \frac{\partial \tilde{z}}{\partial n} \right) dx \, dt$$

$$= \int_{\Sigma_T} \bar{z} \frac{\partial \bar{z}}{\partial n} \, dx \, dt + \int_{\Sigma_T} \tilde{z} \frac{\partial \tilde{z}}{\partial n} \, dx \, dt,$$

which is nonnegative according to Theorem 6.

(only if) Assume first $n = 0$. Then, we can explicitly give the solution of (5.6); it will be enough to give it for $t \leqslant 3L/2 + |y - L/2|$:

$$\begin{aligned}
z(y,t) &= 0 && \text{if } 0 \leqslant t \leqslant L/2 - |y - L/2|,\\
&= \varphi(t-y) && \text{if } y \leqslant t \leqslant L - y,\\
&= \psi(y - L + t) && \text{if } L - y \leqslant t \leqslant y,\\
&= \varphi(t-y) + \psi(y - L + t) && \text{if } L/2 + |y - L/2| \leqslant t\\
& && \quad \leqslant 3L/2 - |y - L/2|,\\
&= \varphi(t-y) + \psi(t - L + y)\\
&\quad - \psi(t - L - y) && \text{if } L + y \leqslant t \leqslant 2L - y,\\
&= \psi(y + t - L) + \varphi(t - y)\\
&\quad - \varphi(y + 2 - 2L) && \text{if } 2L - y \leqslant t \leqslant L + y.
\end{aligned}$$

Therefore, in particular,

$$\left.\frac{\partial z}{\partial n}\right|_{y=0} = -\left.\frac{\partial z}{\partial y}\right|_{y=0} = \begin{cases} \varphi'(t) & \text{if } 0 \leqslant t \leqslant L,\\ \varphi'(t) - 2\psi'(t-L) & \text{if } L \leqslant t \leqslant 2L, \end{cases}$$

$$\left.\frac{\partial z}{\partial n}\right|_{y=L} = \left.\frac{\partial z}{\partial y}\right|_{y=L} = \begin{cases} \psi'(t) & \text{if } 0 \leqslant t \leqslant L,\\ \psi'(t) - 2\varphi'(t-L) & \text{if } L \leqslant t \leqslant 2L. \end{cases}$$

For any $T > L$, and any $m \in \mathbb{N}$, there exists functions φ and ψ such that

$$\int_0^T dt \{\psi(t)(\psi'(t) - 2.1_{(L,t)} \varphi'(t-L)) + \varphi(t)(\varphi'(t) - 2.1_{(L,t)} \psi'(t-L))\}$$

$$\leqslant -m \int_0^T (\varphi(t)^2 + \psi(t)^2) \, dt.$$

In the general case, one replaces $\varphi(t)$ by $\varphi(t)\, a_\varepsilon(x)$, and $\psi(t)$ by $\psi(t)\, a_\varepsilon(x)$, with

$$a_\varepsilon(x) = \varepsilon^{n/2} \exp(-|x|^2 \varepsilon^2).$$

Then, arguing as in Remark 10, one easily proves that $\int_\Sigma z(\partial z/\partial n)\,dx\,dt$ is not positive, and not even bounded from below in

$$L^2(\mathbb{R}^n \times (0, T) \times (\{0\}, \{L\})) = L^2(\Sigma_T). \quad \blacksquare$$

The conclusion we can draw from Theorem 15 is that there is little hope that for general domains the operator A defined by

$$\begin{aligned} \Box u &= 0 && \text{in } \Omega \times (0, T), \\ u(X, 0) = u_t(X, 0) &= 0 && \text{in } \Omega, \\ u &= v && \text{on } \partial\Omega \times (0, T) = \Sigma_T, \end{aligned}$$

and

$$Av = \frac{\partial u}{\partial n}, \qquad n = \text{exterior normal to } \Omega,$$

will be positive, or even bounded from below.

One would expect A to be bounded from below in the exterior of a convex domain. Nevertheless, A is an interesting object which motivates many mathematical studies, and this paper was one more step towards its comprehension.

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